

Cluster algebras of finite type via a Coxeter element and Demazure Crystals of type B,C,D

YUKI KANAKUBO*

Abstract

For a classical group G and a Coxeter element c of the Weyl group, it is known that the coordinate ring $\mathbb{C}[G^{e,c^2}]$ of the double Bruhat cell $G^{e,c^2} := B \cap B_- c^2 B_-$ has a structure of cluster algebra of finite type, where B and B_- are opposite Borel subgroups. In this article, we consider the case G is of type B_r , C_r or D_r and describe all the cluster variables in $\mathbb{C}[G^{e,c^2}]$ as monomial realizations of certain Demazure crystals.

1 Introduction

Fomin and Zelevinsky have invented cluster algebras for the study of total positivity and dual semicanonical bases ([4]). It is a commutative ring generated by so-called “cluster variables”. It is known that the coordinate rings of many algebraic varieties related to semisimple algebraic groups carry cluster algebra structures. For instance, in [1, 8], for simply connected, connected, complex simple algebraic group G and its Weyl group elements $u, v \in W$, it is shown that $\mathbb{C}[G^{u,v}]$ is a cluster algebra, where $G^{u,v} := BuB \cap B_-vB_-$ and B, B_- are opposite Borel subgroups. In [6], it is proved that the coordinate rings $\mathbb{C}[N(w)]$ and $\mathbb{C}[N^w]$ are cluster algebras by using the additive categorification via finite dimensional modules of the preprojective algebras, where $N(w) := N \cap (w^{-1}N_-w)$, $N^w := N \cap (B_-wB_-)$ and N, N_- are unipotent radicals. It is also proved that all the cluster variables are included in the dual semicanonical basis in the coordinate rings.

A cluster algebra is said to be of *finite type* if it has only finitely many cluster variables. In [5], a complete classification of the cluster algebras of finite type are provided. More precisely, they are classified by the set of Cartan matrices up to coefficients. For a fixed Cartan matrix, all the cluster variables are parametrized by the set of “almost positive roots”, which is, a union of all positive roots and negative simple roots corresponding to the Cartan matrix. By this classification, the *type* of each cluster algebra of finite type can be defined as the Cartan-Killing type of the corresponding Cartan matrix. Let $c \in W$ be a Coxeter element such that the length $l(c)$ satisfies $l(c^2) = 2l(c) = 2\text{rank}(G)$. It is known that one can realize a cluster algebra of finite type on the coordinate ring $\mathbb{C}[G^{e,c^2}]$ and its type coincides with the Cartan-Killing type of G [1].

One purpose of our study is to reveal relation between cluster variables of the coordinate rings and Kashiwara’s crystal bases ([13, 14]). The crystal bases

*Division of Mathematics, Sophia University, Kioicho 7-1, Chiyoda-ku, Tokyo 102-8554, Japan: j_chi_sen_you_ky@eagle.sophia.ac.jp

were introduced for combinatorial study of the integrable modules over quantum groups and have many realizations, e.g., tableaux, paths, monomials, etc. In this article, we will treat the *monomial realization*, which is defined in [12, 17].

In [9], we treated the *initial cluster variables* denoted by $\Delta(k; \mathbf{i})$ of $\mathbb{C}[G^{u,e}]$ ($1 \leq k \leq l(u) - r$, $u \in W$ and \mathbf{i} is a reduced word of u) in the case $G = \mathrm{SL}_{r+1}(\mathbb{C})$. We found explicit formulas for $\{\Delta(k; \mathbf{i})\}_{1 \leq k \leq l(u) - r}$, which express them by Laurent polynomials with coefficients 1. We also proved that the set of monomials appearing in $\Delta(k; \mathbf{i})$ coincides with a monomial realization of certain Demazure crystal. In [10], we considered the case G is a classical algebraic group of type B_r , C_r or D_r , and gave explicit formulas for a part of the initial cluster variables $\{\Delta(k; \mathbf{i})\}$ in $\mathbb{C}[G^{u,e}]$. Just as in the case $G = \mathrm{SL}_{r+1}(\mathbb{C})$, the set of monomials appearing in $\Delta(k; \mathbf{i})$ coincides with a monomial realization of certain Demazure crystal. In the both papers, we did not treat all the cluster variables but a part of the cluster variables. On the other hand, in [11], we considered the case $G = \mathrm{SL}_{r+1}(\mathbb{C})$ ($r \geq 3$) and all the cluster variables in the coordinate ring $\mathbb{C}[G^{e,c^2}]$ (c is a Coxeter element). As mentioned above, the algebra $\mathbb{C}[G^{e,c^2}]$ is a cluster algebra of finite type. We described each cluster variable φ as a Laurent polynomial with coefficients 1 and showed that the set of monomials appearing in φ coincides with a monomial realization of the direct sum of certain Demazure crystals.

In this article, we consider the case G is a classical algebraic group of type B_r , C_r or D_r and the coordinate ring $\mathbb{C}[G^{e,c^2}]$, where $c = (s_r \cdots s_2 s_1)$ is a Coxeter element. Our main result is that all the cluster variables in $\mathbb{C}[G^{e,c^2}]$ are described as Laurent polynomials with positive integers, and forgetting the coefficients, the set of monomials appearing in each cluster variable coincides with a monomial realization of the direct sum of certain Demazure crystals.

For example, let us consider the case $G = \mathrm{SO}_5(\mathbb{C})$ (type B_2 algebraic group). Monomial realizations of the crystals $B(\Lambda_1)$ and $B(\Lambda_2)$ of type B_2 are

$$Y_{1,1} \xrightarrow{1} \frac{Y_{2,2}^2}{Y_{2,1}} \xrightarrow{2} \frac{Y_{2,2}}{Y_{3,2}} \xrightarrow{2} \frac{Y_{2,1}}{Y_{3,2}^2} \xrightarrow{1} \frac{1}{Y_{3,1}} \quad \text{and} \quad Y_{1,2} \xrightarrow{2} \frac{Y_{1,1}}{Y_{2,2}} \xrightarrow{1} \frac{Y_{2,2}}{Y_{2,1}} \xrightarrow{2} \frac{1}{Y_{3,2}},$$

respectively. On the other hand, taking a Coxeter element $c = s_2 s_1 \in W$, specific initial cluster variables in $\mathbb{C}[G^{e,c^2}]$ are given by the *generalized minors* $\Delta_{\Lambda_i, s_1 s_2 \Lambda_i}$ ($i = 1, 2$) (see 4.3). Using the biregularly isomorphism $\bar{x}_i^G : H \times (\mathbb{C}^\times)^4 \rightarrow G^{e,c^2}$ ($\mathbf{i} := (2, 1, 2, 1)$) in Proposition 3.4, we have

$$\Delta_{\Lambda_1, s_1 s_2 \Lambda_1} \circ \bar{x}_i^G(a; \mathbf{Y}) = a^{\Lambda_1} (Y_{1,1} + \frac{Y_{2,2}^2}{Y_{2,1}}),$$

$$\Delta_{\Lambda_2, s_1 s_2 \Lambda_2} \circ \bar{x}_i^G(a; \mathbf{Y}) = a^{\Lambda_2} (Y_{1,2} + \frac{Y_{1,1}}{Y_{2,2}} + \frac{Y_{2,2}}{Y_{2,1}}),$$

where we set $a \in H$ and $\mathbf{Y} := (Y_{1,2}, Y_{1,1}, Y_{2,2}, Y_{2,1}) \in (\mathbb{C}^\times)^4$. Comparing with the above crystal graphs of $B(\Lambda_1)$ and $B(\Lambda_2)$, we see that the set of monomials $\{Y_{1,1}, \frac{Y_{2,2}^2}{Y_{2,1}}\}$ (resp. $\{Y_{1,2}, \frac{Y_{1,1}}{Y_{2,2}}, \frac{Y_{2,2}}{Y_{2,1}}\}$) appearing in $\Delta_{\Lambda_1, s_1 s_2 \Lambda_1} \circ \bar{x}_i^G(a; \mathbf{Y})$ (resp. $\Delta_{\Lambda_2, s_1 s_2 \Lambda_2} \circ \bar{x}_i^G(a; \mathbf{Y})$) coincides with the monomial realization of the Demazure crystal $B(\Lambda_1)_{s_1}$ (resp. $B(\Lambda_2)_{s_2 s_1}$) (see 5.2).

All other cluster variables in $\mathbb{C}[G^{e,c^2}]$ are

$$\Delta_{\Lambda_1, s_2 s_1 \Lambda_1} \circ \bar{x}_i^G = a^{\Lambda_1} Y_{2,1}, \quad (\Delta_{s_2 \Lambda_2, s_2 s_1 s_2 \Lambda_2} \cdot \Delta_{\Lambda_2, \Lambda_2}) \circ \bar{x}_i^G = a^{\Lambda_1} Y_{2,2},$$

$$\Delta_{\Lambda_2, s_2 \Lambda_2} \circ \bar{x}_1^G = a^{\Lambda_2} (Y_{1,2} Y_{2,1} + \frac{Y_{1,1} Y_{2,1}}{Y_{2,2}}),$$

$$\begin{aligned} & (\Delta_{s_1 s_2 \Lambda_2, s_2 s_1 s_2 \Lambda_2} \cdot \Delta_{\Lambda_2, s_1 s_2 \Lambda_2} \cdot \Delta_{\Lambda_1, \Lambda_1} - \Delta_{\Lambda_2, \Lambda_2} \cdot \Delta_{\Lambda_2, s_2 s_1 s_2 \Lambda_2}) \circ \bar{x}_1^G \\ &= a^{2\Lambda_2} (Y_{1,2}^2 Y_{2,1} + 2 \frac{Y_{1,1} Y_{1,2} Y_{2,1}}{Y_{2,2}} + \frac{Y_{1,1}^2 Y_{2,1}}{Y_{2,2}^2} + Y_{1,1}), \end{aligned}$$

and the sets of monomials appearing in these Laurent polynomials coincide with monomial realizations of the Demazure crystals $B(\Lambda_1)_e$, $B(\Lambda_2)_e$, $B(\Lambda_1 + \Lambda_2)_{s_2}$ and $B(\Lambda_1 + 2\Lambda_2)_{s_2} \oplus B(\Lambda_1)_e$, respectively. Note that the set of almost positive roots of type B_2 is $\{-\alpha_1, -\alpha_2, \alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. Therefore, the number of the cluster variables in $\mathbb{C}[G^{e, c^2}]$ is 6 from the result of [5].

The article is organized as follows. In section 2, we review the explicit forms of fundamental representations of classical groups. Section 3 is devoted to recall properties of double Bruhat cells. In section 4, after a concise reminder on cluster algebras, we review isomorphisms between the coordinate rings of double Bruhat cells and cluster algebras $\mathcal{A}(\mathbf{i})$. In section 5, we shortly review the monomial realizations of crystal bases. Section 6 presents our main results, which provide a relation between all the cluster variables in $\mathbb{C}[G^{e, c^2}]$ and monomial realizations of Demazure crystals, and we prove them in section 7.

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2 Fundamental representations

First, we review the fundamental representations of the complex simple Lie algebras \mathfrak{g} of type A_r , B_r , C_r , and D_r [16, 18] for calculations of generalized minors (see Subsection 4.3). Let $I := \{1, \dots, r\}$, $A = (a_{ij})_{i,j \in I}$ be the Cartan matrix of \mathfrak{g} , and $(\mathfrak{h}, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I})$ the associated root data satisfying $\alpha_j(h_i) = a_{ij}$, where $\alpha_i \in \mathfrak{h}^*$ is a simple root and $h_i \in \mathfrak{h}$ is a simple co-root. Let $\{\Lambda_i\}_{i \in I}$ be the set of the fundamental weights satisfying $\Lambda_i(h_j) = \delta_{i,j}$, $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$ the weight lattice and $P^* = \bigoplus_{i \in I} \mathbb{Z}h_i$ the dual weight lattice.

2.1 Type A_r

Let $\mathfrak{g} = \mathfrak{sl}(r+1, \mathbb{C})$ be the simple Lie algebra of type A_r . The Cartan matrix $A = (a_{i,j})_{i,j \in I}$ of \mathfrak{g} is as follows:

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathfrak{g} = \langle \mathfrak{h}, e_i, f_i (i \in I) \rangle$, let us describe the vector representation $V(\Lambda_1)$. Set $\mathbf{B}^{(r)} := \{v_i \mid i = 1, 2, \dots, r+1\}$ and define $V(\Lambda_1) := \bigoplus_{v \in \mathbf{B}^{(r)}} \mathbb{C}v$. The weights of v_i ($i = 1, \dots, r+1$) are given by $\text{wt}(v_i) = \Lambda_i - \Lambda_{i-1}$, where $\Lambda_0 = \Lambda_{r+1} = 0$. We define the \mathfrak{g} -action on $V(\Lambda_1)$ as follows: For $i \in I$ and j ($1 \leq j \leq r+1$),

$$hv_j = \langle h, \text{wt}(v_j) \rangle v_j \quad (h \in P^*), \quad f_i v_i = v_{i+1}, \quad e_i v_{i+1} = v_i,$$

and the other actions are trivial.

Let Λ_i be the i -th fundamental weight of type A_r . As is well-known that the fundamental representation $V(\Lambda_i)$ ($1 \leq i \leq r$) is embedded in $\wedge^i V(\Lambda_1)$ with multiplicity free. The explicit form of the highest (resp. lowest) weight vector u_{Λ_i} (resp. v_{Λ_i}) of $V(\Lambda_i)$ is realized in $\wedge^i V(\Lambda_1)$ as follows:

$$u_{\Lambda_i} = v_1 \wedge v_2 \wedge \cdots \wedge v_i, \quad v_{\Lambda_i} = v_{i+1} \wedge v_{i+2} \wedge \cdots \wedge v_{r+1}. \quad (2.1)$$

2.2 Type C_r

Let $\mathfrak{g} = \mathfrak{sp}(2r, \mathbb{C})$ be the simple Lie algebra of type C_r . The Cartan matrix $A = (a_{i,j})_{i,j \in I}$ of \mathfrak{g} is given by

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1 \text{ and } (i, j) \neq (r-1, r), \\ -2 & \text{if } (i, j) = (r-1, r), \\ 0 & \text{otherwise.} \end{cases}$$

Note that α_i ($i \neq r$) are short roots and α_r is the long simple root.

Define the total order on the set $J_C := \{i, \bar{i} | 1 \leq i \leq r\}$ by

$$1 < 2 < \cdots < r-1 < r < \bar{r} < \overline{r-1} < \cdots < \bar{2} < \bar{1}. \quad (2.2)$$

For $\mathfrak{g} = \langle \mathfrak{h}, e_i, f_i (i \in I) \rangle$, let us describe the vector representation $V(\Lambda_1)$. Set $\mathbf{B}^{(r)} := \{v_i, v_{\bar{i}} | i = 1, 2, \dots, r\}$ and define $V(\Lambda_1) := \bigoplus_{v \in \mathbf{B}^{(r)}} \mathbb{C}v$. The weights of $v_i, v_{\bar{i}}$ ($i = 1, \dots, r$) are given by $\text{wt}(v_i) = \Lambda_i - \Lambda_{i-1}$ and $\text{wt}(v_{\bar{i}}) = \Lambda_{i-1} - \Lambda_i$, where $\Lambda_0 = 0$. We define the \mathfrak{g} -action on $V(\Lambda_1)$ as follows:

$$hv_j = \langle h, \text{wt}(v_j) \rangle v_j \quad (h \in P^*, j \in J_C), \quad (2.3)$$

$$f_i v_i = v_{i+1}, \quad f_i v_{\bar{i}+1} = v_{\bar{i}}, \quad e_i v_{i+1} = v_i, \quad e_i v_{\bar{i}} = v_{\bar{i}+1} \quad (1 \leq i < r), \quad (2.4)$$

$$f_r v_r = v_{\bar{r}}, \quad e_r v_{\bar{r}} = v_r, \quad (2.5)$$

and the other actions are trivial.

Let Λ_i be the i -th fundamental weight of type C_r . As is well-known that the fundamental representation $V(\Lambda_i)$ ($1 \leq i \leq r$) is embedded in $\wedge^i V(\Lambda_1)$ with multiplicity free. The explicit form of the highest (resp. lowest) weight vector u_{Λ_i} (resp. v_{Λ_i}) of $V(\Lambda_i)$ is realized in $\wedge^i V(\Lambda_1)$ as follows:

$$u_{\Lambda_i} = v_1 \wedge v_2 \wedge \cdots \wedge v_i, \quad v_{\Lambda_i} = v_{\bar{1}} \wedge v_{\bar{2}} \wedge \cdots \wedge v_{\bar{i}}. \quad (2.6)$$

2.3 Type B_r

Let $\mathfrak{g} = \mathfrak{so}(2r+1, \mathbb{C})$ be the simple Lie algebra of type B_r . The Cartan matrix $A = (a_{i,j})_{i,j \in I}$ of \mathfrak{g} is given by

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1 \text{ and } (i, j) \neq (r, r-1), \\ -2 & \text{if } (i, j) = (r, r-1), \\ 0 & \text{otherwise.} \end{cases}$$

Note that α_i ($i \neq r$) are long roots and α_r is the short simple root.

Define the total order on the set $J_B := \{i, \bar{i} | 1 \leq i \leq r\} \cup \{0\}$ by

$$1 < 2 < \cdots < r-1 < r < 0 < \bar{r} < \overline{r-1} < \cdots < \bar{2} < \bar{1}. \quad (2.7)$$

For $\mathfrak{g} = \langle \mathfrak{h}, e_i, f_i (i \in I) \rangle$, let us describe the vector representation $V(\Lambda_1)$. Set $\mathbf{B}^{(r)} := \{v_i, v_{\bar{i}} | i = 1, 2, \dots, r\} \cup \{v_0\}$ and define $V(\Lambda_1) := \bigoplus_{v \in \mathbf{B}^{(r)}} \mathbb{C}v$. The weights of $v_i, v_{\bar{i}}$ ($i = 1, \dots, r$) and v_0 are as follows:

$$\text{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \quad \text{wt}(v_{\bar{i}}) = \Lambda_{i-1} - \Lambda_i \quad (1 \leq i \leq r-1), \quad (2.8)$$

$$\text{wt}(v_r) = 2\Lambda_r - \Lambda_{r-1}, \quad \text{wt}(v_{\bar{r}}) = \Lambda_{r-1} - 2\Lambda_r, \quad \text{wt}(v_0) = 0,$$

where $\Lambda_0 = 0$. We define the \mathfrak{g} -action on $V(\Lambda_1)$ as follows:

$$hv_j = \langle h, \text{wt}(v_j) \rangle v_j \quad (h \in P^*, j \in J_B), \quad (2.9)$$

$$f_i v_i = v_{i+1}, \quad f_i v_{\bar{i}+1} = v_{\bar{i}}, \quad e_i v_{i+1} = v_i, \quad e_i v_{\bar{i}} = v_{\bar{i}+1} \quad (1 \leq i < r), \quad (2.10)$$

$$f_r v_r = v_0, \quad e_r v_{\bar{r}} = v_0, \quad f_r v_0 = 2v_{\bar{r}}, \quad e_r v_0 = 2v_r, \quad (2.11)$$

and the other actions are trivial.

Let Λ_i ($1 \leq i \leq r-1$) be the i -th fundamental weight of type B_r . Similar to the C_r case, the fundamental representation $V(\Lambda_i)$ is embedded in $\wedge^i V(\Lambda_1)$ with multiplicity free. In $\wedge^i V(\Lambda_1)$, the highest (resp. lowest) weight vector u_{Λ_i} (resp. v_{Λ_i}) of $V(\Lambda_i)$ is realized as the same form as in (2.6).

The fundamental representation $V(\Lambda_r)$ is called the *spin representation*. It can be realized as follows: Set

$$\mathbf{B}_{\text{sp}}^{(r)} := \{(\epsilon_1, \dots, \epsilon_r) | \epsilon_i \in \{+, -\} \quad (i = 1, 2, \dots, r)\}, \quad V_{\text{sp}}^{(r)} := \bigoplus_{v \in \mathbf{B}_{\text{sp}}^{(r)}} \mathbb{C}v,$$

and define the \mathfrak{g} -action on $V_{\text{sp}}^{(r)}$ as follows:

$$h_i(\epsilon_1, \dots, \epsilon_r) = \begin{cases} \frac{\epsilon_i \cdot 1 - \epsilon_{i+1} \cdot 1}{2}(\epsilon_1, \dots, \epsilon_r) & \text{if } i < r, \\ \epsilon_r(\epsilon_1, \dots, \epsilon_r) & \text{if } i = r, \end{cases} \quad (2.12)$$

$$f_i(\epsilon_1, \dots, \epsilon_r) = \begin{cases} (\epsilon_1, \dots, \overset{i}{-}, \overset{i+1}{+}, \dots, \epsilon_r) & \text{if } \epsilon_i = +, \epsilon_{i+1} = -, i \neq r, \\ (\epsilon_1, \dots, \epsilon_{r-1}, \overset{r}{-}) & \text{if } \epsilon_r = +, i = r, \\ 0 & \text{otherwise,} \end{cases} \quad (2.13)$$

$$e_i(\epsilon_1, \dots, \epsilon_r) = \begin{cases} (\epsilon_1, \dots, \overset{i}{+}, \overset{i+1}{-}, \dots, \epsilon_r) & \text{if } \epsilon_i = -, \epsilon_{i+1} = +, i \neq r, \\ (\epsilon_1, \dots, \epsilon_{r-1}, \overset{r}{+}) & \text{if } \epsilon_r = -, i = r, \\ 0 & \text{otherwise.} \end{cases} \quad (2.14)$$

Then the module $V_{\text{sp}}^{(r)}$ is isomorphic to $V(\Lambda_r)$ as a \mathfrak{g} -module.

2.4 Type D_r

Let $\mathfrak{g} = \mathfrak{so}(2r, \mathbb{C})$ be the simple Lie algebra of type D_r . The Cartan matrix $A = (a_{i,j})_{i,j \in I}$ of \mathfrak{g} is given by

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1 \text{ and } (i, j) \neq (r, r-1), (r-1, r), \text{ or } (i, j) = (r-2, r), (r, r-2), \\ 0 & \text{otherwise.} \end{cases}$$

Define the partial order on the set $J_D := \{i, \bar{i} | 1 \leq i \leq r\}$ by

$$1 < 2 < \cdots < r-1 < \frac{r}{r} < \overline{r-1} < \cdots < \bar{2} < \bar{1}. \quad (2.15)$$

Note that there is no order between r and \bar{r} . For $\mathfrak{g} = \langle \mathfrak{h}, e_i, f_i (i \in I) \rangle$, let us describe the vector representation $V(\Lambda_1)$. Set $\mathbf{B}^{(r)} := \{v_i, v_{\bar{i}} | i = 1, 2, \dots, r\}$ and define $V(\Lambda_1) := \bigoplus_{v \in \mathbf{B}^{(r)}} \mathbb{C}v$. The weights of $v_i, v_{\bar{i}}$ ($i \in I$) are as follows:

$$\text{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \quad \text{wt}(v_{\bar{i}}) = \Lambda_{i-1} - \Lambda_i \quad (1 \leq i \leq r-2, i = r), \quad (2.16)$$

$$\text{wt}(v_{r-1}) = \Lambda_r + \Lambda_{r-1} - \Lambda_{r-2}, \quad \text{wt}(v_{\overline{r-1}}) = \Lambda_{r-2} - \Lambda_{r-1} - \Lambda_r,$$

where $\Lambda_0 = 0$. We define the \mathfrak{g} -action on $V(\Lambda_1)$ as follows:

$$hv_j = \langle h, \text{wt}(v_j) \rangle v_j \quad (h \in P^*, j \in J_D), \quad (2.17)$$

$$f_i v_i = v_{i+1}, \quad f_i v_{\bar{i}+1} = v_{\bar{i}}, \quad e_i v_{i+1} = v_i, \quad e_i v_{\bar{i}} = v_{\bar{i}+1} \quad (1 \leq i < r), \quad (2.18)$$

$$f_r v_r = v_{\overline{r-1}}, \quad f_r v_{r-1} = v_{\bar{r}}, \quad e_r v_{\bar{r}} = v_{r-1}, \quad e_r v_{\overline{r-1}} = v_r, \quad (2.19)$$

and the other actions are trivial.

Let Λ_i ($1 \leq i \leq r-2$) be the i -th fundamental weight of type D_r . Similar to the B_r and C_r cases, the fundamental representation $V(\Lambda_i)$ is embedded in $\wedge^i V(\Lambda_1)$ with multiplicity free. In $\wedge^i V(\Lambda_1)$, the highest (resp. lowest) weight vector u_{Λ_i} (resp. v_{Λ_i}) of $V(\Lambda_i)$ is realized as the formula (2.6).

The fundamental representations $V(\Lambda_{r-1})$ and $V(\Lambda_r)$ are also called the *spin representations*. They can be realized as follows: Set

$$\mathbf{B}_{\text{sp}}^{(+,r)} \text{ (resp. } \mathbf{B}_{\text{sp}}^{(-,r)} \text{)} := \{(\epsilon_1, \dots, \epsilon_r) | \epsilon_i \in \{+, -\}, \epsilon_1 \cdots \epsilon_r = + \text{ (resp. } -)\},$$

$$V_{\text{sp}}^{(+,r)} \text{ (resp. } V_{\text{sp}}^{(-,r)} \text{)} := \bigoplus_{v \in \mathbf{B}_{\text{sp}}^{(+,r)} \text{ (resp. } \mathbf{B}_{\text{sp}}^{(-,r)} \text{)}} \mathbb{C}v,$$

and define the \mathfrak{g} -action on $V_{\text{sp}}^{(\pm, r)}$ as follows:

$$h_i(\epsilon_1, \dots, \epsilon_r) = \begin{cases} \frac{\epsilon_i \cdot 1 - \epsilon_{i+1} \cdot 1}{2}(\epsilon_1, \dots, \epsilon_r) & \text{if } i < r, \\ \frac{\epsilon_{r-1} \cdot 1 + \epsilon_r \cdot 1}{2} \epsilon_r(\epsilon_1, \dots, \epsilon_r) & \text{if } i = r, \end{cases} \quad (2.20)$$

$$f_i(\epsilon_1, \dots, \epsilon_r) = \begin{cases} (\epsilon_1, \dots, \overset{i}{-}, \overset{i+1}{+}, \dots, \epsilon_r) & \text{if } \epsilon_i = +, \epsilon_{i+1} = -, i \neq r, \\ (\epsilon_1, \dots, \overset{r-1}{-}, \overset{r}{-}) & \text{if } \epsilon_{r-1} = +, \epsilon_r = +, i = r, \\ 0 & \text{otherwise,} \end{cases} \quad (2.21)$$

$$e_i(\epsilon_1, \dots, \epsilon_r) = \begin{cases} (\epsilon_1, \dots, \overset{i}{+}, \overset{i+1}{-}, \dots, \epsilon_r) & \text{if } \epsilon_i = -, \epsilon_{i+1} = +, i \neq r, \\ (\epsilon_1, \dots, \overset{r-1}{+}, \overset{r}{+}) & \text{if } \epsilon_{r-1} = -, \epsilon_r = -, i = r, \\ 0 & \text{otherwise.} \end{cases} \quad (2.22)$$

Then the module $V_{\text{sp}}^{(+,r)}$ (resp. $V_{\text{sp}}^{(-,r)}$) is isomorphic to $V(\Lambda_r)$ (resp. $V(\Lambda_{r-1})$) as a \mathfrak{g} -module.

3 Factorization theorem

In this section, we shall introduce double Bruhat cells $G^{u,v}$ and their properties [2, 3]. For $l \in \mathbb{Z}_{>0}$, we set $[1, l] := \{1, 2, \dots, l\}$.

3.1 Double Bruhat cells

Let G be a classical algebraic group over \mathbb{C} , B and B_- be two opposite Borel subgroups in G , $N \subset B$ and $N_- \subset B_-$ be their unipotent radicals, $H := B \cap B_-$ a maximal torus. We set $\mathfrak{g} := \text{Lie}(G)$ with the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Let e_i, f_i ($i \in [1, r]$) be the generators of $\mathfrak{n}, \mathfrak{n}_-$. For $i \in [1, r]$ and $t \in \mathbb{C}$, we set

$$x_i(t) := \exp(te_i), \quad y_i(t) := \exp(tf_i). \quad (3.1)$$

Let $W := \langle s_i | i = 1, \dots, r \rangle$ be the Weyl group of \mathfrak{g} , where $\{s_i\}$ are the simple reflections. We identify the Weyl group W with $\text{Norm}_G(H)/H$. An element

$$\overline{s_i} := x_i(-1)y_i(1)x_i(-1) \quad (3.2)$$

is in $\text{Norm}_G(H)$, which is a representative of $s_i \in W = \text{Norm}_G(H)/H$ [18]. For $u \in W$, let $u = s_{i_1} \cdots s_{i_n}$ be its reduced expression. Then we write $\overline{u} = \overline{s_{i_1}} \cdots \overline{s_{i_n}}$, call $l(u) := n$ the length of u . We have two kinds of Bruhat decompositions of G as follows:

$$G = \coprod_{u \in W} B\overline{u}B = \coprod_{u \in W} B_- \overline{u} B_-.$$

Then, for $u, v \in W$, we define the *double Bruhat cell* $G^{u,v}$ as follows:

$$G^{u,v} := B\overline{u}B \cap B_- \overline{v} B_-.$$

Definition 3.1. Let $v = s_{j_n} \cdots s_{j_1}$ be a reduced expression of $v \in W$ ($j_n, \dots, j_1 \in [1, r]$). Then the finite sequence $\mathbf{i} := (j_n, \dots, j_1)$ is called a *reduced word* for v .

For example, the sequence $(3, 2, 1, 3, 2, 1)$ is a reduced word of the element $s_3 s_2 s_1 s_3 s_2 s_1$ of the Weyl group of type B_3 or C_3 . In this paper, we mainly treat double Bruhat cells of the form $G^{e,v} := B \cap B_- \overline{v} B_-$.

3.2 Factorization theorem

In this subsection, we shall introduce isomorphisms between double Bruhat cells $G^{e,v}$ and $H \times (\mathbb{C}^\times)^{l(v)}$. For $i \in [1, r]$ and $t \in \mathbb{C}^\times$, we set $\alpha_i^\vee(t) := t^{h_i}$.

For a sequence $\mathbf{i} = (i_1, \dots, i_n)$ ($i_1, \dots, i_n \in [1, r]$), we define a map $x_{\mathbf{i}}^G : H \times \mathbb{C}^n \rightarrow G$ as

$$x_{\mathbf{i}}^G(a; t_1, \dots, t_n) := a \cdot x_{i_1}(t_1) \cdots x_{i_n}(t_n). \quad (3.3)$$

Theorem 3.2. [2, 3] For $v \in W$ and its reduced word \mathbf{i} , the map $x_{\mathbf{i}}^G$ is a biregular isomorphism from $H \times (\mathbb{C}^\times)^{l(v)}$ to a Zariski open subset of $G^{e,v}$.

For $\mathbf{i} = (i_1, \dots, i_n)$ ($i_1, \dots, i_n \in [1, r]$), we define a map $\bar{x}_{\mathbf{i}}^G : H \times (\mathbb{C}^\times)^n \rightarrow G^{e,v}$ as

$$\bar{x}_{\mathbf{i}}^G(a; t_1, \dots, t_n) = ax_{i_1}(t_1)\alpha_{i_1}^\vee(t_1)x_{i_2}(t_2)\alpha_{i_2}^\vee(t_2) \cdots x_{i_n}(t_n)\alpha_{i_n}^\vee(t_n),$$

where $a \in H$ and $(t_1, \dots, t_n) \in (\mathbb{C}^\times)^n$.

Now, let G be a classical algebraic group of type B_r , C_r or D_r , and $c \in W$ be a Coxeter element such that a reduced word \mathbf{i} of c^2 can be written as

$$\mathbf{i} = (r, r-1, \dots, 2, 1)^2. \quad (3.4)$$

Remark 3.3. In the rest of the paper, we use double indexed variables $Y_{s,j}$ ($s \in \mathbb{Z}$, $j \in [1, r]$). If we see the variables $Y_{s,0}$, $Y_{s,j}$ ($r+1 \leq j$) then we understand $Y_{s,0} = Y_{s,j} = 1$. For example, if $l = 1$ then $Y_{s,l-1} = 1$.

Proposition 3.4. In the above setting, the map $\bar{x}_{\mathbf{i}}^G$ is a biregular isomorphism between $H \times (\mathbb{C}^\times)^{2r}$ and a Zariski open subset of G^{e,c^2} .

[Proof.]

Let us prove this proposition in the case G is type C_r . In the case G is type B_r or D_r , we can prove it in the same way.

In this proof, we use the notation

$$\mathbf{Y} := (Y_{1,r}, Y_{1,r-1}, \dots, Y_{1,1}, Y_{2,r}, Y_{2,r-1}, \dots, Y_{2,1}),$$

for variables instead of $(t_1, \dots, t_{2r}) \in (\mathbb{C}^\times)^{2r}$.

We define a map $\phi : H \times (\mathbb{C}^\times)^{2r} \rightarrow H \times (\mathbb{C}^\times)^{2r}$,

$$\phi(a; \mathbf{Y}) = (\Phi_H(a; \mathbf{Y}); \Phi_{1,r}(\mathbf{Y}), \dots, \Phi_{1,1}(\mathbf{Y}), \Phi_{2,r}(\mathbf{Y}), \dots, \Phi_{2,2}(\mathbf{Y}), \Phi_{2,1}(\mathbf{Y})),$$

as

$$\Phi_H(a; \mathbf{Y}) := a \cdot \prod_{i=1}^r \prod_{j=1}^2 \alpha_i^\vee(Y_{j,i}), \quad (3.5)$$

and for $l \in \{1, 2, \dots, r\}$,

$$\Phi_{1,l}(\mathbf{Y}) := \begin{cases} \frac{Y_{1,l-1}Y_{2,l-1}Y_{2,l+1}}{Y_{1,l}Y_{2,l}^2} & \text{if } l < r, \\ \frac{Y_{1,r-1}^2Y_{2,r-1}^2}{Y_{1,r}Y_{2,r}^2} & \text{if } l = r, \end{cases} \quad (3.6)$$

$$\Phi_{2,l}(\mathbf{Y}) := \begin{cases} \frac{Y_{2,l-1}}{Y_{2,l}} & \text{if } l < r, \\ \frac{Y_{2,r-1}^2}{Y_{2,r}} & \text{if } l = r. \end{cases} \quad (3.7)$$

Note that ϕ is a biregular isomorphism since we can construct the inverse map $\psi : H \times (\mathbb{C}^\times)^{2r} \rightarrow H \times (\mathbb{C}^\times)^{2r}$,

$$\psi(a; \mathbf{Y}) = (\Psi_H(a; \mathbf{Y}); \Psi_{1,r}(\mathbf{Y}), \dots, \Psi_{1,1}(\mathbf{Y}), \Psi_{2,r}(\mathbf{Y}), \dots, \Psi_{2,1}(\mathbf{Y}))$$

of ϕ as follows:

$$\Psi_{2,l}(\mathbf{Y}) := \begin{cases} (Y_{2,1} \cdots Y_{2,l-1} Y_{2,l})^{-1} & \text{if } l < r, \\ (Y_{2,1} \cdots Y_{2,r-2} Y_{2,r-1})^{-2} Y_{2,r}^{-1} & \text{if } l = r, \end{cases}$$

and $\Psi_{1,l}(\mathbf{Y})$ is defined inductively as

$$\Psi_{1,l}(\mathbf{Y}) := \begin{cases} \frac{\Psi_{1,l-1}(\mathbf{Y}) \Psi_{2,l-1}(\mathbf{Y}) \Psi_{2,l+1}(\mathbf{Y})}{\Psi_{2,l}^2(\mathbf{Y}) Y_{1,l}} & \text{if } l < r, \\ \frac{\Psi_{1,r-1}^2(\mathbf{Y}) \Psi_{2,r-1}^2(\mathbf{Y})}{\Psi_{2,r}^2(\mathbf{Y}) Y_{1,r}} & \text{if } l = r, \end{cases}$$

and

$$\Psi_H(a; \mathbf{Y}) := a \cdot \left(\prod_{i=1}^r \prod_{j=1}^2 \alpha_i^\vee(\Psi_{j,i}(\mathbf{Y})) \right)^{-1}.$$

Then, the map ψ is the inverse map of ϕ .

Let us prove

$$\bar{x}_1^G(a; \mathbf{Y}) = (x_1^G \circ \phi)(a; \mathbf{Y}),$$

which implies that $\bar{x}_1^G : H \times (\mathbb{C}^\times)^{2r} \rightarrow G^{e,c^2}$ is a biregular isomorphism by Theorem 3.2. First, it is known that for $1 \leq i, j \leq r$ and $s, t \in \mathbb{C}^\times$,

$$\alpha_i^\vee(s) x_j(t) = x_j(s^{a_{i,j}} t) \alpha_i^\vee(s), \quad (3.8)$$

where $(a_{i,j})_{i,j \in I}$ is the Cartan matrix of type C_r . On the other hand, it follows from the definition (3.3) of x_1^G and (3.5) that

$$\begin{aligned} (x_1^G \circ \phi)(a; \mathbf{Y}) &= a \cdot \left(\prod_{i=1}^r \prod_{s=1}^2 \alpha_i^\vee(Y_{s,i}) \right) \times x_r(\Phi_{1,r}(\mathbf{Y})) \cdots x_1(\Phi_{1,1}(\mathbf{Y})) \\ &\quad \times x_r(\Phi_{2,r}(\mathbf{Y})) \cdots x_2(\Phi_{2,2}(\mathbf{Y})) x_1(\Phi_{2,1}(\mathbf{Y})). \end{aligned}$$

For each l ($1 \leq l < r$), we can move

$$\prod_{i=1}^l \alpha_i^\vee(Y_{1,i}) \prod_{i=1}^r \alpha_i^\vee(Y_{2,i})$$

to the right of $x_l(\Phi_{1,l}(\mathbf{Y}))$ by using the relations (3.8):

$$\begin{aligned} &\left(\prod_{i=1}^l \alpha_i^\vee(Y_{1,i}) \prod_{i=1}^r \alpha_i^\vee(Y_{2,i}) \right) x_l(\Phi_{1,l}(\mathbf{Y})) \\ &= x_l(\Phi_{1,l}(\mathbf{Y})) \frac{Y_{1,l}^2 Y_{2,l}^2}{Y_{1,l-1} Y_{2,l-1} Y_{2,l+1}} \prod_{i=1}^l \alpha_i^\vee(Y_{1,i}) \prod_{i=1}^r \alpha_i^\vee(Y_{2,i}) \\ &= x_l(Y_{1,l}) \prod_{i=1}^l \alpha_i^\vee(Y_{1,i}) \prod_{i=1}^r \alpha_i^\vee(Y_{2,i}). \end{aligned}$$

In the same way, we see that

$$\left(\prod_{i=1}^r \alpha_i^\vee(Y_{1,i}) \prod_{i=1}^r \alpha_i^\vee(Y_{2,i}) \right) x_r(\Phi_{1,r}(\mathbf{Y})) = x_r(Y_{1,r}) \left(\prod_{i=1}^r \alpha_i^\vee(Y_{1,i}) \prod_{i=1}^r \alpha_i^\vee(Y_{2,i}) \right).$$

Similarly, for $1 \leq l \leq r$, we can also move $\prod_{i=1}^l \alpha_i^\vee(Y_{2,i})$ to the right of $x_l(\Phi_{2,l}(\mathbf{Y}))$:

$$\left(\prod_{i=1}^l \alpha_i^\vee(Y_{2,i}) \right) x_l(\Phi_{2,l}(\mathbf{Y})) = x_l(Y_{2,l}) \left(\prod_{i=1}^l \alpha_i^\vee(Y_{2,i}) \right).$$

Thus, we get

$$\begin{aligned} (x_i^G \circ \phi)(a; \mathbf{Y}) &= a \cdot x_r(Y_{1,r}) \alpha_r^\vee(Y_{1,r}) \cdots x_1(Y_{1,1}) \alpha_1^\vee(Y_{1,1}) \\ &\quad x_r(Y_{2,r}) \alpha_r^\vee(Y_{2,r}) \cdots x_2(Y_{2,2}) \alpha_2^\vee(Y_{2,2}) x_1(Y_{2,1}) \alpha_1^\vee(Y_{2,1}) = \bar{x}_i^G(a; \mathbf{Y}). \end{aligned}$$

□

4 Cluster algebras and generalized minors

Following [1, 3, 4, 7], we review the definitions of cluster algebras and their generators called cluster variables. It is known that the coordinate rings of double Bruhat cells have cluster algebra structures, and generalized minors are their initial cluster variables [8]. We will refer to a relation between cluster variables on double Bruhat cells and crystal bases in Sect.6.

We set $[1, l] := \{1, 2, \dots, l\}$ and $[-1, -l] := \{-1, -2, \dots, -l\}$ for $l \in \mathbb{Z}_{>0}$. For $n, m \in \mathbb{Z}_{>0}$, let $x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}$ be commuting variables and $\mathcal{F} := \mathbb{C}(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})$ be the field of rational functions.

4.1 Cluster algebras of geometric type

In this subsection, we recall the definitions of cluster algebras. Let $\tilde{B} = (b_{ij})_{1 \leq i \leq n+m, 1 \leq j \leq n}$ be an $(n+m) \times n$ integer matrix. The *principal part* B of \tilde{B} is obtained from \tilde{B} by deleting the last m rows. For \tilde{B} and $k \in [1, n]$, the new $(n+m) \times n$ integer matrix $\mu_k(\tilde{B}) = (b'_{ij})$ is defined by

$$b'_{ij} := \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}b_{kj} + b_{ik}b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

One calls $\mu_k(\tilde{B})$ the *matrix mutation* in direction k of \tilde{B} . If there exists a positive integer diagonal matrix D such that DB is skew symmetric, we say B is *skew symmetrizable*. Then we also say \tilde{B} is skew symmetrizable. It is easily verified that if \tilde{B} is skew symmetrizable then $\mu_k(\tilde{B})$ is also skew symmetrizable[7, Proposition 3.6]. We can also verify that $\mu_k \mu_k(\tilde{B}) = \tilde{B}$. Define $\mathbf{x} := (x_1, \dots, x_{n+m})$ and we call the pair (\mathbf{x}, \tilde{B}) *initial seed* and x_1, \dots, x_{n+m} *initial cluster variables*. For $k \in [1, n]$, a new cluster variable x'_k is defined by the following *exchange relation*.

$$x_k x'_k = \prod_{1 \leq i \leq n+m, b_{ik} > 0} x_i^{b_{ik}} + \prod_{1 \leq i \leq n+m, b_{ik} < 0} x_i^{-b_{ik}}. \quad (4.1)$$

Let $\mu_k(\mathbf{x})$ be the set of variables obtained from \mathbf{x} by replacing x_k by x'_k . Ones call the pair $(\mu_k(\mathbf{x}), \mu_k(\tilde{B}))$ the *mutation* in direction k of the seed (\mathbf{x}, \tilde{B}) and denote it by $\mu_k((\mathbf{x}, \tilde{B}))$.

Now, we can repeat this process of mutation and obtain a set of seeds inductively. Hence, each seed consists of an $(n + m)$ -tuple of variables and a matrix. Ones call this $(n + m)$ -tuple and matrix *cluster* and *exchange matrix* respectively. Variables in cluster is called *cluster variables*. In particular, the variables x_{n+1}, \dots, x_{n+m} are called *frozen cluster variables*.

Definition 4.1. [3, 7] Let \tilde{B} be an integer matrix whose principal part is skew symmetrizable and $\Sigma = (\mathbf{x}, \tilde{B})$ a seed. We set $\mathbb{A} := \mathbb{Z}[x_{n+1}^{\pm 1}, \dots, x_{n+m}^{\pm 1}]$. The cluster algebra (of geometric type) $\mathcal{A} = \mathcal{A}(\Sigma)$ over \mathbb{A} associated with seed Σ is defined as the \mathbb{A} -subalgebra of \mathcal{F} generated by all cluster variables in all seeds which can be obtained from Σ by sequences of mutations.

4.2 Cluster algebra $\mathcal{A}(\mathbf{i})$

Let G be a classical algebraic group, $\mathfrak{g} := \text{Lie}(G)$ and $A = (a_{i,j})$ be its Cartan matrix. In Definition 3.1, we define a reduced word $\mathbf{i} = (j_n, \dots, j_2, j_1)$ for an element v of Weyl group W . In this subsection, we define a cluster algebra $\mathcal{A}(\mathbf{i})$, which is obtained from \mathbf{i} . It satisfies that $\mathcal{A}(\mathbf{i}) \otimes \mathbb{C}$ is isomorphic to the coordinate ring $\mathbb{C}[G^{e,v}]$ of the double Bruhat cell [1]. Let j_k ($k \in [1, n]$) be the k -th index of \mathbf{i} from the right. Let us add r additional entries j_{-r}, \dots, j_{-1} at the beginning of \mathbf{i} by setting $j_{-t} = -t$ ($t \in [1, r]$).

For $l \in [1, n]$, we denote by l^- the largest index $k \in [1, n]$ such that $k < l$ and $j_l = j_k$. If $l \in [-1, -r]$, let l^- be the largest index $k \in [1, n]$ such that $|j_l| = |j_k|$. For example, if $[-1, -3] \cup \mathbf{i} = (-3, -2, -1, 3, 2, 1, 3, 2, 1)$ then, $(-1)^- = 4$, $(-2)^- = 5$, $(-3)^- = 6$, $4^- = 1$, $5^- = 2$, $6^- = 3$, and 3^- , 2^- , 1^- are not defined. We define a set $e(\mathbf{i})$ as

$$e(\mathbf{i}) := \{k \in [1, n] \mid k^- \text{ is well-defined}\}.$$

Following [1], we define a directed graph $\Gamma_{\mathbf{i}}$ as follows. The vertices of $\Gamma_{\mathbf{i}}$ are the variables x_k ($k \in [-1, -r] \cup [1, n]$). For two vertices x_k ($k \in [-1, -r] \cup [1, n]$) and x_l ($l \in e(\mathbf{i})$) with either $l < k$ or $k \in [-1, -r]$, there exists an arrow $x_k \rightarrow x_l$ (resp. $x_l \rightarrow x_k$) if and only if $l = k^-$ (resp. $l^- < k^- < l$ and $a_{|j_k|, |j_l|} < 0$). Next, let us define a matrix $\tilde{B} = \tilde{B}(\mathbf{i})$.

Definition 4.2. Let $\tilde{B}(\mathbf{i})$ be an integer matrix with rows labelled by all the indices in $[-1, -r] \cup [1, n]$ and columns labelled by all the indices in $e(\mathbf{i})$. For $k \in [-1, -r] \cup [1, n]$ and $l \in e(\mathbf{i})$, an entry b_{kl} of $\tilde{B}(\mathbf{i})$ is determined as follows: If there exists an arrow $x_k \rightarrow x_l$ (resp. $x_l \rightarrow x_k$) in $\Gamma_{\mathbf{i}}$, then

$$b_{kl} := \begin{cases} 1 \text{ (resp. } -1) & \text{if } |j_k| = |j_l|, \\ -a_{|j_k|, |j_l|} \text{ (resp. } a_{|j_k|, |j_l|}) & \text{if } |j_k| \neq |j_l|. \end{cases}$$

If there exist no arrows between k and l , we set $b_{kl} = 0$. The principal part $B(\mathbf{i})$ of $\tilde{B}(\mathbf{i})$ is the submatrix $(b_{i,j})_{i,j \in e(\mathbf{i})}$.

Proposition 4.3. [1] $\tilde{B}(\mathbf{i})$ is skew symmetrizable.

Definition 4.4. [1] We set $\mathbf{x} = (x_i)_{i \in [-1, -r] \cup [1, n]}$ and define the cluster algebra $\mathcal{A}(\mathbf{i})$ over $\mathbb{Z}[x_i^{\pm 1} \mid i \in [-1, -r] \cup [1, n] \setminus e(\mathbf{i})]$ as $\mathcal{A}(\mathbf{i}) := \mathcal{A}(\mathbf{x}, \tilde{B}(\mathbf{i}))$.

In this definition, we use the notation x_i ($i \in [-1, -r] \cup [1, n] \setminus e(\mathbf{i})$) for frozen cluster variables instead of x_{n+1}, \dots, x_{n+m} in Definition 4.1.

4.3 Generalized minors

Set $\mathcal{A}(\mathbf{i})_{\mathbb{C}} := \mathcal{A}(\mathbf{i}) \otimes \mathbb{C}$. It is known that the coordinate ring $\mathbb{C}[G^{e,v}]$ of the double Bruhat cell is isomorphic to $\mathcal{A}(\mathbf{i})_{\mathbb{C}}$ (Theorem 4.6). To describe this isomorphism explicitly, we need generalized minors.

We set $G_0 := N_- H N$, and let $x = [x]_- [x]_0 [x]_+$ with $[x]_- \in N_-$, $[x]_0 \in H$, $[x]_+ \in N$ be the corresponding decomposition.

Definition 4.5. For $i \in [1, r]$ and $w, w' \in W$, the *generalized minor* $\Delta_{w' \Lambda_i, w \Lambda_i}$ is a regular function on G whose restriction to the open set $\overline{w'} G_0 \overline{w}^{-1}$ is given by $\Delta_{w' \Lambda_i, w \Lambda_i}(x) = ([\overline{w'}^{-1} x \overline{w}]_0)^{\Lambda_i}$. Here, Λ_i is the i -th fundamental weight and \overline{w} is the one we defined in (3.2).

The generalized minor $\Delta_{w' \Lambda_i, w \Lambda_i}$ depends on $w' \Lambda_i, w \Lambda_i$ and does not depend on w', w . By definition, for $a \in H$, $x \in G$, $w \in W$, $i, j \in I$ and $t \in \mathbb{C}$,

$$\Delta_{\Lambda_i, w \Lambda_i}(ax) = a^{\Lambda_i} \Delta_{\Lambda_i, w \Lambda_i}(x), \quad \Delta_{\Lambda_i, \Lambda_i}(xx_j(t)) = \Delta_{\Lambda_i, \Lambda_i}(x), \quad (4.2)$$

where $x_j(t) \in N$ is the one in (3.1).

Let $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ be the anti-involution

$$\omega(e_i) = f_i, \quad \omega(f_i) = e_i, \quad \omega(h) = h,$$

and extend it to G by setting $\omega(x_i(c)) = y_i(c)$, $\omega(y_i(c)) = x_i(c)$ and $\omega(t) = t$ ($t \in H$). Here, $x_i(t)$ and $y_i(t)$ were defined in (3.1). One can calculate the generalized minors as follows. There exists a \mathfrak{g} (or G)-invariant bilinear form on the finite-dimensional irreducible \mathfrak{g} -module $V(\lambda)$ such that

$$\langle au, v \rangle = \langle u, \omega(a)v \rangle, \quad (u, v \in V(\lambda), a \in \mathfrak{g} \text{ (or } G)).$$

For $g \in G$, we have the following simple fact:

$$\Delta_{\Lambda_i, \Lambda_i}(g) = \langle gu_{\Lambda_i}, u_{\Lambda_i} \rangle,$$

where u_{Λ_i} is a properly normalized highest weight vector in $V(\Lambda_i)$. Hence, for $w, w' \in W$, we have

$$\Delta_{w' \Lambda_i, w \Lambda_i}(g) = \Delta_{\Lambda_i}(\overline{w'}^{-1} g \overline{w}) = \langle g \overline{w} \cdot u_{\Lambda_i}, \overline{w'} \cdot u_{\Lambda_i} \rangle. \quad (4.3)$$

Note that $\omega(\overline{s}_i^{\pm}) = \overline{s}_i^{\mp}$.

4.4 Cluster algebras on double Bruhat cells

For a reduced expression $v = s_{j_n} s_{j_{n-1}} \cdots s_{j_1} \in W$ and $k \in [1, n]$, we set

$$v_{>k} = v_{>k}(\mathbf{i}) := s_{j_1} s_{j_2} \cdots s_{j_{n-k}}. \quad (4.4)$$

For $k \in [1, n]$, we define $\Delta(k; \mathbf{i})(x) := \Delta_{\Lambda_{j_k}, v_{>n-k+1}\Lambda_{j_k}}(x)$, and for $k \in [-1, -r]$, $\Delta(k; \mathbf{i})(x) := \Delta_{\Lambda_{|k|}, v^{-1}\Lambda_{|k|}}(x)$. Finally, we set $F(\mathbf{i}) := \{\Delta(k; \mathbf{i})(x) | k \in [-1, -r] \cup [1, n]\}$. It is known that the set $F(\mathbf{i})$ is an algebraically independent generating set for the field of rational functions $\mathbb{C}(G^{e,v})$ [3, Theorem 1.12]. Then, we have the following.

Theorem 4.6. [1, 6, 8] *The isomorphism of fields $\varphi : \mathcal{F} \rightarrow \mathbb{C}(G^{e,v})$ defined by $\varphi(x_k) = \Delta(k; \mathbf{i})$ ($k \in [-1, -r] \cup [1, n]$) restricts to an isomorphism of algebras $\mathcal{A}(\mathbf{i})_{\mathbb{C}} \rightarrow \mathbb{C}[G^{e,v}]$.*

Example 4.7. *Let G be a classical algebraic group of type B_r , C_r or D_r , $v = c^2$ be the square of a Coxeter element such that whose reduced word \mathbf{i} is written as in (3.4). Then for $k \in [1, r]$, we have $j_{r+k} = j_k = k$ and the isomorphism is given by*

$$\begin{aligned} x_{-k} &\mapsto \Delta_{\Lambda_k, c^{-2}\Lambda_k} = \Delta_{\Lambda_k, (s_1 s_2 \cdots s_r)^2 \Lambda_k} = \Delta_{\Lambda_k, (s_1 s_2 \cdots s_r)(s_1 s_2 \cdots s_k) \Lambda_k} = \Delta_{\Lambda_k, c_{>r-k}^2 \Lambda_k}, \\ x_{r+k} &\mapsto \Delta_{\Lambda_k, c_{>r-k+1}^2 \Lambda_k} = \Delta_{\Lambda_k, (s_1 s_2 \cdots s_r)(s_1 s_2 \cdots s_{k-1}) \Lambda_k} \\ &= \Delta_{\Lambda_k, (s_1 s_2 \cdots s_k) \Lambda_k} = \Delta_{\Lambda_k, c_{>2r-k}^2 \Lambda_k}, \\ x_k &\mapsto \Delta_{\Lambda_k, c_{>2r-k+1}^2 \Lambda_k} = \Delta_{\Lambda_k, (s_1 s_2 \cdots s_{k-1}) \Lambda_k} = \Delta_{\Lambda_k, \Lambda_k}. \end{aligned}$$

4.5 Finite type

Let \mathcal{S} be the set of seeds of a cluster algebra \mathcal{A} . If \mathcal{S} is finite, then \mathcal{A} is said to be *finite type*. In this subsection, we shall review the cluster algebras of finite type [5].

Let $B = (b_{ij})$ be an integer square matrix. The *Cartan counter part* of B is a generalized Cartan matrix $A = A(B) = (a_{i,j})$ defined as follows:

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -|b_{i,j}| & \text{if } i \neq j. \end{cases}$$

Theorem 4.8. [5] *For a cluster algebra \mathcal{A} with the set \mathcal{S} of seeds, the following statements are equivalent:*

- (i) *The cluster algebra \mathcal{A} is of finite type.*
- (ii) *There exists a seed $\Sigma = (\mathbf{y}, \tilde{B})$ such that $\mathcal{A} = \mathcal{A}(\Sigma)$ and $A(B)$ is a Cartan matrix of finite type, where B is the principal part of \tilde{B} .*
- (iii) *Let $(\mathbf{y}', \tilde{B}')$ be an arbitrary seed in \mathcal{S} and $(b_{i,j})$ be the principal part of \tilde{B}' . Then $|b_{i,j} b_{j,i}| \leq 3$.*

By this theorem, we can define the *type* of each cluster algebra of finite type mirroring the Cartan-Killing classification.

Let Φ be the root system associated with a Cartan matrix, with the set of simple roots $\Pi = \{\alpha_i | i \in I\}$ and the set of positive roots $\Phi_{>0}$. The set of *almost positive roots* is defined as $\Phi_{\geq -1} := \Phi_{>0} \cup -\Pi$.

Theorem 4.9. [5]

- (i) *For a cluster algebra \mathcal{A} of finite type, the number of the cluster variables included in \mathcal{A} is equal to $|\Phi_{\geq -1}|$. Here, Φ is the root system associated with the Cartan matrix of the same type as \mathcal{A} .*

- (ii) Let $c \in W$ be a Coxeter element of a classical algebraic group G whose length $l(c)$ satisfies $l(c^2) = 2l(c) = 2\text{rank}(G)$. Then the coordinate ring $\mathbb{C}[G^{e,c^2}]$ has a structure of cluster algebra of finite type under the isomorphism in Theorem 4.6, and its type is the Cartan-Killing type of G .

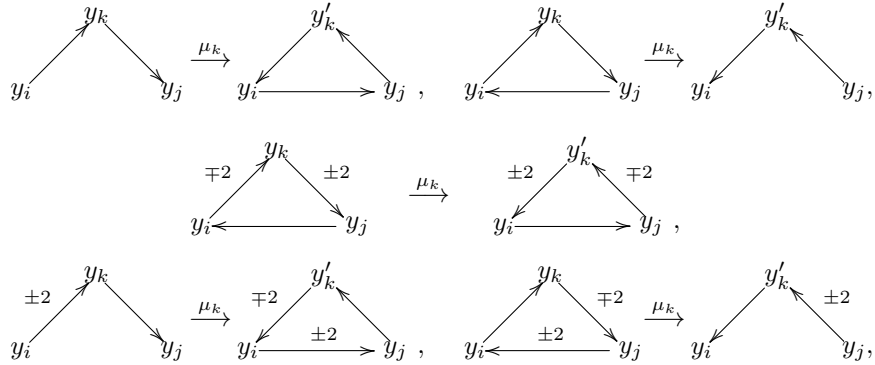
Next, we define the following graph, which is a similar notion to the *weighted graph* introduced in [5].

Definition 4.10. Let $\Sigma = (\mathbf{y}, \tilde{B})$ be a seed with $\mathbf{y} = (y_i)_{i \in [-1, -r] \cup [1, n]}$ and an $(r+n) \times |e(\mathbf{i})|$ -skew symmetrizable matrix $\tilde{B} = (b_{i,j})$ which satisfies $b_{i,j} \in \{-2, -1, 0, 1, 2\}$ (the rows of \tilde{B} are labelled by $[-1, -r] \cup [1, n]$ as above), where $y_i = x_i$ for $i \in [-1, -r] \cup [1, n] \setminus e(\mathbf{i})$. We suppose that if $i, j \in e(\mathbf{i})$ then $|b_{i,j}b_{j,i}| \leq 3$. We define $\Gamma(\Sigma)$ as the labelled directed graph whose vertices are $y_{-r}, \dots, y_{-1}, y_1, \dots, y_n$, and whose arrows and its labels are determined as follows: For $i, j \in e(\mathbf{i})$, there exists the arrow $y_i \xrightarrow{2} y_j$ (resp. $y_j \xrightarrow{-2} y_i$) if and only if $b_{i,j} = 2$ and $b_{j,i} = -1$ (resp. $b_{i,j} = -2$ and $b_{j,i} = 1$). Further, there exists the arrow $y_i \rightarrow y_j$ if and only if $b_{i,j} = 1$ and $b_{j,i} = -1$. For $i \in [-1, -r] \cup [1, n] \setminus e(\mathbf{i})$ and $j \in e(\mathbf{i})$, there exists the arrow $y_i \xrightarrow{2} y_j$ (resp. $y_j \xrightarrow{-2} y_i$) if and only if $b_{i,j} = 2$ (resp. $b_{i,j} = -2$). Further, there exists the arrow $y_i \rightarrow y_j$ (resp. $y_j \rightarrow y_i$) if and only if $b_{i,j} = 1$ (resp. $b_{i,j} = -1$). We call the graph $\Gamma(\Sigma)$ *mutation diagram* of Σ . We understand the arrows $y_i \rightarrow y_j$ ($i, j \in [-1, -r] \cup [1, n]$) have the labels 1 and do not denote it.

Note that the graph $\Gamma((\mathbf{x}, \tilde{B}(\mathbf{i})))$ is obtained from $\Gamma_{\mathbf{i}}$ by labelling properly.

Lemma 4.11. [5, 7] Let $\Sigma = (\mathbf{y}, \tilde{B})$ be a seed as in Definition 4.10. For $k \in e(\mathbf{i})$, the graph $\Gamma(\mu_k(\Sigma))$ has vertices $y_{-r}, \dots, y_{-1}, y_1, \dots, y'_k, \dots, y_n$, and edges or subgraphs of $\Gamma(\Sigma)$ are transformed to those of $\Gamma(\mu_k(\Sigma))$ by μ_k as follows:

- (1) If $y_i \rightarrow y_k$ (resp. $y_k \rightarrow y_i$) in $\Gamma(\Sigma)$ then $y'_k \rightarrow y_i$ (resp. $y_i \rightarrow y'_k$) in $\Gamma(\mu_k(\Sigma))$. If $y_i \xrightarrow{\pm 2} y_k$ in $\Gamma(\Sigma)$ then $y'_k \xrightarrow{\mp 2} y_i$ in $\Gamma(\mu_k(\Sigma))$.
- (2) For $i, j \in [-1, -r] \cup [1, n]$, we suppose that either i or j (or both) belong to $e(\mathbf{i})$. Then

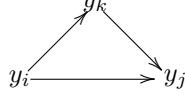


where the arrow $y_i \xrightarrow{-2} y_j$ with $i \notin e(\mathbf{i})$, $j \in e(\mathbf{i})$ implies the arrow $y_i \rightarrow y_j$, and the arrow $y_i \xrightarrow{2} y_j$ with $i \in e(\mathbf{i})$, $j \notin e(\mathbf{i})$ implies the arrow $y_i \rightarrow y_j$.

- (3) If two vertices y_i and y_j are not connected via a two-arrow oriented path going through y_k in $\Gamma(\Sigma)$, the arrows between y_i and y_j and their labels remain unchanged by μ_k .

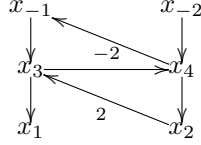
We will use this lemma in Sect 7.

Remark 4.12. In the above lemma, we do not mention to several subgraphs. For example, the subgraph

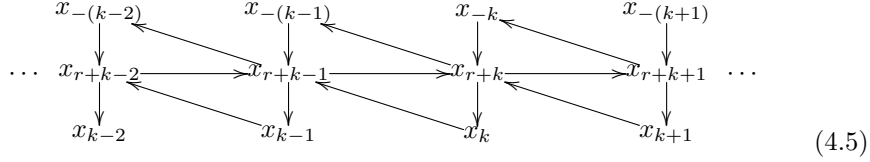


is not mentioned. But we will not treat these graphs in this article.

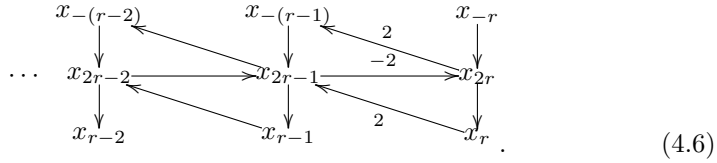
Example 4.13. Let us consider the case $G = \mathrm{SO}_5(\mathbb{C})$ (type B_2) and $\mathbf{i} = (2, 1, 2, 1)$. The graph $\Gamma((\mathbf{x}, \tilde{B}(\mathbf{i})))$ is described as



In general, let us consider the case $G = \mathrm{SO}_{2r+1}(\mathbb{C})$ (type B_r) and \mathbf{i} is the sequence in (3.4). For k ($1 \leq k \leq r-2$), vertices and arrows around the vertex x_{r+k} in the graph $\Gamma((\mathbf{x}, \tilde{B}(\mathbf{i})))$ are



Vertices and arrows around the vertex x_{2r} in the graph $\Gamma((\mathbf{x}, \tilde{B}(\mathbf{i})))$ are



5 Monomial realizations and Demazure crystals

In Sect.6, we shall describe the cluster variables in a cluster algebra of finite type in terms of the *monomial realizations* of Demazure crystals. Let us recall the notion of crystal base and its monomial realization in this section. Let \mathfrak{g} be a complex simple Lie algebra and $I = \{1, 2, \dots, r\}$ the index set.

5.1 Monomial realizations of crystals

In this subsection, we review the monomial realizations of crystals [12, 14, 17]. First, let us recall the crystals.

Definition 5.1. [13] A *crystal* associated with a Cartan matrix A is a set B together with the maps $\text{wt} : B \rightarrow P$, $\tilde{e}_i, \tilde{f}_i : B \cup \{0\} \rightarrow B \cup \{0\}$ and $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}$, $i \in I$, satisfying some properties.

We call $\{\tilde{e}_i, \tilde{f}_i\}$ the *Kashiwara operators*. Let $U_q(\mathfrak{g})$ be the quantum enveloping algebra [13] associated with a Cartan matrix A , that is, $U_q(\mathfrak{g})$ has generators $\{e_i, f_i, h_i \mid i \in I\}$ over $\mathbb{C}(q)$ satisfying some relations, where q is an indeterminate. Let $V(\lambda)$ ($\lambda \in P^+ = \oplus_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i$) be the finite dimensional irreducible representation of $U_q(\mathfrak{g})$ which has the highest weight vector v_λ , and $B(\lambda)$ be the crystal base of $V(\lambda)$. The crystal base $B(\lambda)$ has a crystal structure.

Let us introduce monomial realizations which realize each element of $B(\lambda)$ as a certain Laurent monomial. First, fix a cyclic sequence of the indices $\cdots (i_1, i_2, \dots, i_r)(i_1, i_2, \dots, i_r) \cdots$ such that $\{i_1, i_2, \dots, i_r\} = I$. And we can associate this sequence with a set of integers $p = (p_{j,i})_{j,i \in I, j \neq i}$ such that

$$p_{i_a, i_b} = \begin{cases} 1 & \text{if } a < b, \\ 0 & \text{if } a > b. \end{cases}$$

Second, for doubly-indexed variables $\{Y_{s,i} \mid i \in I, s \in \mathbb{Z}\}$, we define the set of monomials

$$\mathcal{Y} := \left\{ Y = \prod_{s \in \mathbb{Z}, i \in I} Y_{s,i}^{\zeta_{s,i}} \mid \zeta_{s,i} \in \mathbb{Z}, \zeta_{s,i} = 0 \text{ except for finitely many } (s,i) \right\}.$$

Finally, we define maps $\text{wt} : \mathcal{Y} \rightarrow P$, $\varepsilon_i, \varphi_i : \mathcal{Y} \rightarrow \mathbb{Z}$, $i \in I$. For $Y = \prod_{s \in \mathbb{Z}, i \in I} Y_{s,i}^{\zeta_{s,i}} \in \mathcal{Y}$,

$$\text{wt}(Y) := \sum_{i,s} \zeta_{s,i} \Lambda_i, \quad \varphi_i(Y) := \max \left\{ \sum_{k \leq s} \zeta_{k,i} \mid s \in \mathbb{Z} \right\}, \quad \varepsilon_i(Y) := \varphi_i(Y) - \text{wt}(Y)(h_i). \quad (5.1)$$

We set

$$A_{s,i} := Y_{s,i} Y_{s+1,i} \prod_{j \neq i} Y_{s+p_{j,i},j}^{a_{j,i}} \quad (5.2)$$

and define the Kashiwara operators as follows

$$\tilde{f}_i Y = \begin{cases} A_{n_{f_i},i}^{-1} Y & \text{if } \varphi_i(Y) > 0, \\ 0 & \text{if } \varphi_i(Y) = 0, \end{cases} \quad \tilde{e}_i Y = \begin{cases} A_{n_{e_i},i} Y & \text{if } \varepsilon_i(Y) > 0, \\ 0 & \text{if } \varepsilon_i(Y) = 0, \end{cases}$$

where

$$n_{f_i} := \min \left\{ n \mid \varphi_i(Y) = \sum_{k \leq n} \zeta_{k,i} \right\}, \quad n_{e_i} := \max \left\{ n \mid \varphi_i(Y) = \sum_{k \leq n} \zeta_{k,i} \right\}.$$

Then the following theorem holds:

Theorem 5.2. [12, 17]

- (i) For the set $p = (p_{j,i})$ as above, $(\mathcal{Y}, \text{wt}, \varphi_i, \varepsilon_i, \tilde{f}_i, \tilde{e}_i)_{i \in I}$ is a crystal. When we emphasize p , we write \mathcal{Y} as $\mathcal{Y}(p)$.
- (ii) If a monomial $Y \in \mathcal{Y}(p)$ satisfies $\varepsilon_i(Y) = 0$ for all $i \in I$, then the connected component containing Y is isomorphic to $B(\text{wt}(Y))$.

5.2 Demazure crystals

For $w \in W$ and $\lambda \in P^+$, a *Demazure crystal* $B(\lambda)_w \subset B(\lambda)$ is inductively defined as follows.

Definition 5.3. Let u_λ be the highest weight vector of $B(\lambda)$. For the identity element e of W , we set $B(\lambda)_e := \{u_\lambda\}$. For $w \in W$, if $s_i w < w$,

$$B(\lambda)_w := \{\tilde{f}_i^k b \mid k \geq 0, b \in B(\lambda)_{s_i w}, \tilde{e}_i b = 0\} \setminus \{0\}.$$

Theorem 5.4. [15] For $w \in W$, let $w = s_{i_1} \cdots s_{i_n}$ be an arbitrary reduced expression. Let u_λ be the highest weight vector of $B(\lambda)$. Then

$$B(\lambda)_w = \{\tilde{f}_{i_1}^{a(1)} \cdots \tilde{f}_{i_n}^{a(n)} u_\lambda \mid a(1), \dots, a(n) \in \mathbb{Z}_{\geq 0}\} \setminus \{0\}.$$

Example 5.5. Let us consider the case of type C_2 and cyclic sequence is $(2, 1)$. In the notation of (5.2), we can write

$$A_{s,i} = \begin{cases} \frac{Y_{s,1} Y_{s+1,1}}{Y_{s+1,2}} & \text{if } i = 1, \\ \frac{Y_{s,2} Y_{s+1,2}}{Y_{s,1}^2} & \text{if } i = 2. \end{cases}$$

In general, if each factor of a monomial $Y \in \mathcal{Y}$ has non-negative degree, then $\varepsilon_i(Y) = 0$ for all $i \in I = \{1, 2\}$. Therefore, we have $\varepsilon_i(Y_{1,1}) = 0$. Hence, we can consider the monomial realization of crystal base $B(\Lambda_1)$ such that the highest weight vector in $B(\Lambda_1)$ is realized by $Y_{1,1}$:

$$Y_{1,1} \xrightarrow{\tilde{f}_1} \frac{Y_{2,2}}{Y_{2,1}} \xrightarrow{\tilde{f}_2} \frac{Y_{2,1}}{Y_{3,2}} \xrightarrow{\tilde{f}_1} \frac{1}{Y_{3,1}}. \quad (5.3)$$

Similarly, we get the monomial realization of crystal base $B(\Lambda_2)$ such that the highest weight vector in $B(\Lambda_2)$ is realized by $Y_{1,2}$:

$$Y_{1,2} \xrightarrow{\tilde{f}_2} \frac{Y_{1,1}^2}{Y_{2,2}} \xrightarrow{\tilde{f}_1} \frac{Y_{1,1}}{Y_{2,1}} \xrightarrow{\tilde{f}_1} \frac{Y_{2,2}}{Y_{2,1}^2} \xrightarrow{\tilde{f}_2} \frac{1}{Y_{3,2}}. \quad (5.4)$$

Example 5.6. Let us consider the case of type B_3 and cyclic sequence is $(3, 2, 1)$. In the notation of (5.2), we can write

$$A_{s,i} = \begin{cases} \frac{Y_{s,1} Y_{s+1,1}}{Y_{s+1,2}} & \text{if } i = 1, \\ \frac{Y_{s,2} Y_{s+1,2}}{Y_{s,1}^2} & \text{if } i = 2, \\ \frac{Y_{s,3} Y_{s+1,3}}{Y_{s,2}} & \text{if } i = 3. \end{cases}$$

We can consider the monomial realization of crystal base $B(\Lambda_1)$ such that the highest weight vector in $B(\Lambda_1)$ is realized by $Y_{1,1}$:

$$Y_{1,1} \xrightarrow{\tilde{f}_1} \frac{Y_{2,2}}{Y_{2,1}} \xrightarrow{\tilde{f}_2} \frac{Y_{3,3}^2}{Y_{3,2}} \xrightarrow{\tilde{f}_3} \frac{Y_{3,3}}{Y_{4,3}} \xrightarrow{\tilde{f}_3} \frac{Y_{3,2}}{Y_{4,3}^2} \xrightarrow{\tilde{f}_2} \frac{Y_{3,1}}{Y_{4,2}} \xrightarrow{\tilde{f}_1} \frac{1}{Y_{4,1}}.$$

6 Cluster variables and crystals

Let G be a classical algebraic group of type B_r , C_r or D_r . In this section, we describe the cluster variables on a double Bruhat cell by the total sum of monomial realizations of Demazure crystals. In the rest of the paper, we only treat the Coxeter element $c \in W$ such that a reduced word \mathbf{i} of c^2 can be written as (3.4). Let j_k be the k -th index of \mathbf{i} from the right, which implies $j_k = j_{r+k} = k$ ($1 \leq k \leq r$). We shall consider the monomial realization associated with the sequence $(r, \dots, 2, 1)$ (Sect.5.1).

Let $\mathbf{V} := ((\varphi_{\mathbf{V}})_{2r}, \dots, (\varphi_{\mathbf{V}})_{r+1}, (\varphi_{\mathbf{V}})_r, \dots, (\varphi_{\mathbf{V}})_1, (\varphi_{\mathbf{V}})_{-r}, \dots, (\varphi_{\mathbf{V}})_{-1})$, where $(\varphi_{\mathbf{V}})_k \in \mathbb{C}[G^{e,c^2}]$ are defined as follows:

$$(\varphi_{\mathbf{V}})_k = \begin{cases} \Delta_{\Lambda_{j_k, c^2} > 2r-k} \Lambda_{j_k} & \text{if } 1 \leq k \leq 2r, \\ \Delta_{\Lambda_{|k|}, \Lambda_{|k|}} & \text{if } -r \leq k \leq -1. \end{cases} \quad (6.1)$$

By Theorem 4.6, Example 4.7 and Theorem 4.9, we can regard $\mathbb{C}[G^{e,c^2}]$ as a cluster algebra of finite type and \mathbf{V} as its initial cluster. Moreover, $(\varphi_{\mathbf{V}})_{2r}, \dots, (\varphi_{\mathbf{V}})_{r+1}$ and $(\varphi_{\mathbf{V}})_{-r}, \dots, (\varphi_{\mathbf{V}})_{-1}$ are frozen cluster variables.

Using these notation, we can rewrite the graph (4.5) of type B_r as

$$\begin{array}{ccccccc} & (\varphi_{\mathbf{V}})_{r+k-2} & & (\varphi_{\mathbf{V}})_{r+k-1} & & (\varphi_{\mathbf{V}})_{r+k} & & (\varphi_{\mathbf{V}})_{r+k+1} \\ & \swarrow & & \swarrow & & \swarrow & & \swarrow \\ \cdots & (\varphi_{\mathbf{V}})_{k-2} & \longrightarrow & (\varphi_{\mathbf{V}})_{k-1} & \longrightarrow & (\varphi_{\mathbf{V}})_k & \longrightarrow & (\varphi_{\mathbf{V}})_{k+1} \cdots \\ & \searrow & & \searrow & & \searrow & & \searrow \\ & (\varphi_{\mathbf{V}})_{-(k-2)} & & (\varphi_{\mathbf{V}})_{-(k-1)} & & (\varphi_{\mathbf{V}})_{-k} & & (\varphi_{\mathbf{V}})_{-(k+1)} \end{array} \quad (6.2)$$

Similarly, we can also rewrite (4.6) by using these notation:

$$\begin{array}{ccccc} & (\varphi_{\mathbf{V}})_{2r-2} & & (\varphi_{\mathbf{V}})_{2r-1} & & (\varphi_{\mathbf{V}})_{2r} \\ & \swarrow & & \swarrow & & \swarrow \\ \cdots & (\varphi_{\mathbf{V}})_{r-2} & \longrightarrow & (\varphi_{\mathbf{V}})_{r-1} & \xrightarrow{-2} & (\varphi_{\mathbf{V}})_r \\ & \searrow & & \searrow & & \searrow \\ & (\varphi_{\mathbf{V}})_{-(r-2)} & & (\varphi_{\mathbf{V}})_{-(r-1)} & \xrightarrow{2} & (\varphi_{\mathbf{V}})_{-r} \end{array} \quad (6.3)$$

In the rest of the paper, when we write a cluster in $\mathbb{C}[G^{e,c^2}]$, we drop frozen variables. For example, $\mathbf{V} = ((\varphi_{\mathbf{V}})_r, \dots, (\varphi_{\mathbf{V}})_1)$. We will order the cluster variables $(\varphi_{\mathbf{V}})_1, \dots, (\varphi_{\mathbf{V}})_r$ from the right in \mathbf{V} as above, and let μ_k denote the mutation of the k -th cluster variable from the right. For a cluster \mathbf{T} in $\mathbb{C}[G^{e,c^2}]$, let $(\varphi_{\mathbf{T}})_k$ denote the k -th (non-frozen) cluster variable from the right:

$$\mathbf{T} := ((\varphi_{\mathbf{T}})_r, \dots, (\varphi_{\mathbf{T}})_1).$$

Each cluster variable is a regular function on G^{e,c^2} . On the other hand, by Proposition 3.4, it can be seen as a function on $H \times (\mathbb{C}^\times)^{2r}$. Then, let us consider the following change of variables:

Definition 6.1. Along (3.4), we set the variables $\mathbf{Y} \in (\mathbb{C}^\times)^{2r}$ as

$$\mathbf{Y} := (Y_{1,r}, Y_{1,r-1}, \dots, Y_{1,2}, Y_{1,1}, Y_{2,r}, Y_{2,r-1}, \dots, Y_{2,2}, Y_{2,1}). \quad (6.4)$$

Then for $a \in H$ and cluster \mathbf{T} in $\mathbb{C}[G^{e,c^2}]$, we define

$$(\varphi_{\mathbf{T}}^G)_k(a; \mathbf{Y}) := (\varphi_{\mathbf{T}})_k \circ \bar{x}_1^G(a; \mathbf{Y}), \quad (1 \leq k \leq r),$$

where \bar{x}_1^G is as in Proposition 3.4.

Example 6.2. Let us consider the case $G = \mathrm{Sp}_4(\mathbb{C})$ (type C_2) and $\mathbf{i} = (2, 1, 2, 1)$. In the above setting, for $k \in [1, r]$,

$$(\varphi_V^G)_k(a; \mathbf{Y}) = (\varphi_V)_k \circ \bar{x}_i^G(a; \mathbf{Y}).$$

The definition (6.1) says $(\varphi_V)_1 = \Delta_{\Lambda_1, c_{>3}^2 \Lambda_1} = \Delta_{\Lambda_1, s_1 \Lambda_1}$. Using the bilinear form (4.3), it follows from the actions (2.8), (2.10), (2.11) on the fundamental representation that

$$\begin{aligned} (\varphi_V^G)_1(a; \mathbf{Y}) &= (\varphi_V)_1 \circ \bar{x}_i^G(a; \mathbf{Y}) \\ &= \langle a \cdot x_2(Y_{1,2}) \alpha_2^\vee(Y_{1,2}) x_1(Y_{1,1}) \alpha_1^\vee(Y_{1,1}) x_2(Y_{2,2}) \alpha_2^\vee(Y_{2,2}) x_1(Y_{2,1}) \alpha_1^\vee(Y_{2,1}) \bar{s}_1 v_1, v_1 \rangle \\ &= a^{\Lambda_1} \langle v_2, \alpha_1^\vee(Y_{2,1}) y_1(Y_{2,1}) \alpha_2^\vee(Y_{2,2}) y_2(Y_{2,2}) \alpha_1^\vee(Y_{1,1}) y_1(Y_{1,1}) \alpha_2^\vee(Y_{1,2}) y_2(Y_{1,2}) v_1 \rangle \\ &= a^{\Lambda_1} \langle v_2, \alpha_1^\vee(Y_{2,1}) y_1(Y_{2,1}) \alpha_2^\vee(Y_{2,2}) y_2(Y_{2,2}) (Y_{1,1} v_1 + v_2) \rangle \\ &= a^{\Lambda_1} \langle v_2, \left(Y_{1,1} Y_{2,1} v_1 + (Y_{1,1} + \frac{Y_{2,2}}{Y_{2,1}}) v_2 + Y_{2,1} v_{\bar{2}} + v_{\bar{1}} \right) \rangle \\ &= a^{\Lambda_1} (Y_{1,1} + \frac{Y_{2,2}}{Y_{2,1}}). \end{aligned}$$

Comparing with (5.3), the set of monomials $\{Y_{1,1}, \frac{Y_{2,2}}{Y_{2,1}}\}$ coincides with the monomial realization of the Demazure crystal $B(\Lambda_1)_{s_1}$ in Example 5.5 (see Theorem 5.4).

Note that the value of $(\varphi_V^G)_1(a; \mathbf{Y})$ was reduced to the calculation of the coefficient of v_2 in $\alpha_1^\vee(Y_{2,1}) y_1(Y_{2,1}) \alpha_2^\vee(Y_{2,2}) y_2(Y_{2,2}) \alpha_1^\vee(Y_{1,1}) y_1(Y_{1,1}) \alpha_2^\vee(Y_{1,2}) y_2(Y_{1,2}) v_1$. Similarly, the value of $(\varphi_V^G)_2(a; \mathbf{Y})$ was reduced to the calculation of the coefficient of $v_2 \wedge v_{\bar{1}}$ in $\alpha_1^\vee(Y_{2,1}) y_1(Y_{2,1}) \cdots \alpha_2^\vee(Y_{1,2}) y_2(Y_{1,2}) v_1 \wedge v_2$. Hence, to calculate $(\varphi_V^G)_2(a; \mathbf{Y})$, we need

$$\begin{aligned} &\alpha_1^\vee(Y_{2,1}) y_1(Y_{2,1}) \alpha_2^\vee(Y_{2,2}) y_2(Y_{2,2}) \alpha_1^\vee(Y_{1,1}) y_1(Y_{1,1}) \alpha_2^\vee(Y_{1,2}) y_2(Y_{1,2}) v_2 \\ &= \alpha_1^\vee(Y_{2,1}) y_1(Y_{2,1}) \alpha_2^\vee(Y_{2,2}) y_2(Y_{2,2}) (\frac{Y_{1,2}}{Y_{1,1}} v_2 + Y_{1,1} v_{\bar{2}} + v_{\bar{1}}) \\ &= \frac{Y_{1,2} Y_{2,2}}{Y_{1,1} Y_{2,1}} v_2 + (\frac{Y_{1,1} Y_{2,1}}{Y_{2,2}} + \frac{Y_{1,2} Y_{2,1}}{Y_{1,1}}) v_{\bar{2}} + (\frac{1}{Y_{2,1}} + \frac{Y_{1,1}}{Y_{2,2}} + \frac{Y_{1,2}}{Y_{1,1}}) v_{\bar{1}}. \end{aligned}$$

Just as in the case of $(\varphi_V^G)_1(a; \mathbf{Y})$, one get

$$\begin{aligned} (\varphi_V^G)_2(a; \mathbf{Y}) &= (\varphi_V)_2 \circ x_i^G \circ \phi(a; \mathbf{Y}) \\ &= a^{\Lambda_2} \langle v_2 \wedge v_{\bar{1}}, \alpha_1^\vee(Y_{2,1}) y_1(Y_{2,1}) \cdots \alpha_2^\vee(Y_{1,2}) y_2(Y_{1,2}) v_1 \wedge v_2 \rangle \\ &= a^{\Lambda_2} \langle v_2 \wedge v_{\bar{1}}, \left((Y_{1,1} + \frac{Y_{2,2}}{Y_{2,1}}) v_2 \wedge (\frac{1}{Y_{2,1}} + \frac{Y_{1,1}}{Y_{2,2}} + \frac{Y_{1,2}}{Y_{1,1}}) v_{\bar{1}} \right) + v_{\bar{1}} \wedge \frac{Y_{1,2} Y_{2,2}}{Y_{1,1} Y_{2,1}} v_2 \rangle \\ &= a^{\Lambda_2} (Y_{1,2} + \frac{Y_{1,1}^2}{Y_{2,2}} + 2 \frac{Y_{1,1}}{Y_{2,1}} + \frac{Y_{2,2}}{Y_{2,1}^2}). \end{aligned}$$

Comparing with (5.4), the set of monomials $\{Y_{1,2}, \frac{Y_{1,1}^2}{Y_{2,2}}, \frac{Y_{1,1}}{Y_{2,1}}, \frac{Y_{2,2}}{Y_{2,1}^2}\}$ is equal to the monomial realization of the Demazure crystal $B(\Lambda_2)_{s_1 s_2}$ in Example 5.6.

The following theorems are our main results, which mean relations between the cluster variables in $\mathbb{C}[G^{e,c^2}]$ and Demazure crystals.

Theorem 6.3. *Let $G = \mathrm{SO}_{2r+1}(\mathbb{C})$ be the classical algebraic group of type B_r . The cluster variables in $\mathbb{C}[G^{e,c^2}]$ are the total sums of monomial realizations of certain Demazure crystals. More precisely, each cluster variable is described as follows:*

(i) For $k \in [1, r]$, we obtain

$$\begin{aligned} (\varphi_V^G)_k(a; \mathbf{Y}) &= a^{\Lambda_k} Y_{1,k} (1 + A_{1,k}^{-1} + A_{1,k}^{-1} A_{1,k-1}^{-1} + \cdots + A_{1,k}^{-1} A_{1,k-1}^{-1} \cdots A_{1,1}^{-1}) \\ &= a^{\Lambda_k} \sum_{b \in B(\Lambda_k)_{c^2}{}_{>2r-k}} \mu(b), \end{aligned} \quad (6.5)$$

where $\mu : B(\Lambda_k) \rightarrow \mathcal{Y}$ is the monomial realization of $B(\Lambda_k)$ such that the highest weight vector in $B(\Lambda_k)$ is realized by $Y_{1,k} \in \mathcal{Y}$. We also obtain

$$(\varphi_{\mu_k \cdots \mu_{r-1} \mu_r(\mathbf{V})}^G)_k(a; \mathbf{Y}) = \begin{cases} a^{\Lambda_{k-1} + \Lambda_{r-1}} Y_{2,k} = a^{\Lambda_{k-1} + \Lambda_{r-1}} \sum_{b \in B(\Lambda_k)_e} \mu'(b) & \text{if } k < r, \\ a^{\Lambda_{r-1}} Y_{2,r} = a^{\Lambda_{r-1}} \sum_{b \in B(\Lambda_k)_e} \mu'(b) & \text{if } k = r, \end{cases}$$

where μ' is the monomial realization of $B(\Lambda_k)$ such that the highest weight vector in $B(\Lambda_k)$ is realized by $Y_{2,k} \in \mathcal{Y}$.

(ii) For k and l with $1 \leq k \leq l \leq r-2$,

$$\begin{aligned} &(\varphi_{\mu_l \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_l(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1} + \Lambda_{l+1}} Y_{2,k} Y_{1,l+1} (1 + A_{1,l+1}^{-1} + A_{1,l+1}^{-1} A_{1,l}^{-1} + \cdots + A_{1,l+1}^{-1} A_{1,l}^{-1} \cdots A_{1,k+1}^{-1}) \end{aligned} \quad (6.6)$$

$$= a^{\Lambda_{k-1} + \Lambda_{l+1}} \sum_{b \in B(\Lambda_k + \Lambda_{l+1})_{s_{k+1} s_{k+2} \cdots s_{l+1}}} \mu(b), \quad (6.7)$$

where μ is the monomial realization of $B(\Lambda_k + \Lambda_{l+1})$ such that the highest weight vector in $B(\Lambda_k + \Lambda_{l+1})$ is realized by $Y_{2,k} Y_{1,l+1} \in \mathcal{Y}$.

(iii) For $k \in [1, r-1]$, we obtain

$$\begin{aligned} &(\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_{r-1}(a; \mathbf{Y}) \\ &= a^{2\Lambda_r + \Lambda_{k-1}} \left(Y_{1,r}^2 Y_{2,k} + 2Y_{1,r}^2 Y_{2,k} A_{1,r}^{-1} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \right. \\ &\quad \left. + Y_{1,r}^2 Y_{2,k} A_{1,r}^{-2} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1})^2 \right. \\ &\quad \left. + Y_{1,k-1} Y_{1,r-1} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,1}^{-1}) \right) \\ &= a^{\Lambda_{k-1} + 2\Lambda_r} \sum_{b \in B(\Lambda_k + 2\Lambda_r)_{s_{k+1} s_{k+2} \cdots s_r}} C(b) \mu(b) \\ &\quad + a^{\Lambda_{k-1} + 2\Lambda_r} \sum_{b \in B(\Lambda_{k-1} + \Lambda_{r-1})_{s_1 \cdots s_{k-1} s_{k+1} s_{k+2} \cdots s_{r-1}}} \mu'(b), \end{aligned} \quad (6.8)$$

where μ is the monomial realization of $B(\Lambda_k + 2\Lambda_r)$ such that the highest weight vector is realized by $Y_{2,k} Y_{1,r}^2 \in \mathcal{Y}$, μ' is the monomial realization of $B(\Lambda_{k-1} + \Lambda_{r-1})$ such that the highest weight vector is realized by

$Y_{1,k-1}Y_{1,r-1} \in \mathcal{Y}$, and $C(b)$ are some positive integers. We also obtain

$$\begin{aligned} & (\varphi_{\mu_r \mu_{r-1} \dots \mu_{k+1} \mu_k}^G(\mathbf{v}))_r(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1} + \Lambda_r} Y_{2,k} Y_{1,r} (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-1}^{-1} + \dots + A_{1,r}^{-1} A_{1,r-1}^{-1} \dots A_{1,k+1}^{-1}), \end{aligned} \quad (6.9)$$

$$= a^{\Lambda_{k-1} + \Lambda_r} \sum_{b \in B(\Lambda_k + \Lambda_r)_{s_{k+1} s_{k+2} \dots s_r}} \mu''(b), \quad (6.10)$$

where μ'' is the monomial realization of $B(\Lambda_k + \Lambda_r)$ such that the highest weight vector is realized by $Y_{2,k} Y_{1,r} \in \mathcal{Y}$.

(iv) For j and k with $1 \leq j < k \leq r-2$,

$$\begin{aligned} & (\varphi_{\mu_j \dots \mu_{k-2} \mu_{k-1} \mu_r \mu_{r-1} \dots \mu_{k+1} \mu_k}^G(\mathbf{v}))_j(a; \mathbf{Y}) = a^{\Lambda_{j-1} + \Lambda_{k-1} + 2\Lambda_r} (Y_{2,k} Y_{2,j} Y_{1,r}^2 \\ & + 2 \frac{Y_{2,k} Y_{2,j} Y_{1,r-1} Y_{1,r}}{Y_{2,r}} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \dots + A_{1,r-1}^{-1} \dots A_{1,k+1}^{-1})) \end{aligned} \quad (6.11)$$

$$\begin{aligned} & + a^{\Lambda_{j-1} + \Lambda_{k-1} + 2\Lambda_r} \frac{Y_{2,k} Y_{2,j} Y_{1,r-1}^2}{Y_{2,r}^2} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \dots + A_{1,r-1}^{-1} \dots A_{1,k+1}^{-1}) \\ & \quad \times (1 + A_{1,r-1}^{-1} + \dots + A_{1,r-1}^{-1} \dots A_{1,j+1}^{-1}) \\ &= a^{\Lambda_{j-1} + \Lambda_{k-1} + 2\Lambda_r} \sum_{b \in B(\Lambda_j + \Lambda_k + 2\Lambda_r)_{s_{k+1} s_{k+2} \dots s_r}} C(b) \mu(b) \\ & \quad + a^{\Lambda_{j-1} + \Lambda_{k-1} + 2\Lambda_r} \sum_{b \in B(\Lambda_j + \Lambda_{k-1} + \Lambda_{r-1})_{s_{j+1} \dots s_{k-1} s_{k+1} s_{k+2} \dots s_{r-1}}} \mu'(b), \end{aligned} \quad (6.12)$$

where μ is the monomial realization of $B(\Lambda_j + \Lambda_k + 2\Lambda_r)$ such that the highest weight vector is realized by $Y_{2,j} Y_{2,k} Y_{1,r}^2 \in \mathcal{Y}$, μ' is the monomial realization of $B(\Lambda_j + \Lambda_{k-1} + \Lambda_{r-1})$ such that the highest weight vector is realized by $Y_{2,j} Y_{1,k-1} Y_{1,r-1} \in \mathcal{Y}$, and $C(b)$ are some positive integers.

Theorem 6.4. Let $G = \mathrm{Sp}_{2r}(\mathbb{C})$ be the classical algebraic group of type C_r .

(i) For $k \in [1, r]$, we obtain

$$(\varphi_{\mathbf{v}}^G)_k(a; \mathbf{Y}) = a^{\Lambda_k} \sum_{b \in B(\Lambda_k)_{e_{>2r-k}^2}} C(b) \mu(b),$$

where $\mu : B(\Lambda_k) \rightarrow \mathcal{Y}$ is the monomial realization of $B(\Lambda_k)$ such that the highest weight vector in $B(\Lambda_k)$ is realized by $Y_{1,k} \in \mathcal{Y}$, and $C(b)$ are some positive integers. We also obtain

$$(\varphi_{\mu_k \dots \mu_{r-1} \mu_r}^G(\mathbf{v}))_k(a; \mathbf{Y}) = a^{\Lambda_{k-1} + \Lambda_{r-1}} Y_{2,k} = a^{\Lambda_{k-1} + \Lambda_{r-1}} \sum_{b \in B(\Lambda_k)_e} \mu'(b),$$

where μ' is the monomial realization of $B(\Lambda_k)$ such that the highest weight vector in $B(\Lambda_k)$ is realized by $Y_{2,k} \in \mathcal{Y}$.

(ii) For k and l with $1 \leq k \leq l \leq r-2$,

$$\begin{aligned} & (\varphi_{\mu_l \dots \mu_{k+1} \mu_k}^G(\mathbf{v}))_l(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1} + \Lambda_{l+1}} Y_{2,k} Y_{1,l+1} (1 + A_{1,l+1}^{-1} + A_{1,l+1}^{-1} A_{1,l}^{-1} + \dots + A_{1,l+1}^{-1} A_{1,l}^{-1} \dots A_{1,k+1}^{-1}) \end{aligned} \quad (6.13)$$

$$= a^{\Lambda_{k-1} + \Lambda_{l+1}} \sum_{b \in B(\Lambda_k + \Lambda_{l+1})_{s_{k+1} s_{k+2} \dots s_{l+1}}} \mu(b), \quad (6.14)$$

where μ is the monomial realization of $B(\Lambda_k + \Lambda_{l+1})$ such that the highest weight vector in $B(\Lambda_k + \Lambda_{l+1})$ is realized by $Y_{2,k}Y_{1,l+1} \in \mathcal{Y}$.

(iii) For $k \in [1, r-1]$, we obtain

$$\begin{aligned}
& (\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{v}))_{r-1}(a; \mathbf{Y}) \\
&= a^{\Lambda_{k-1} + \Lambda_r} \left(Y_{2,k}Y_{1,r} + Y_{2,k}Y_{1,r}A_{1,r}^{-1}(1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1})^2 \right. \\
&+ Y_{1,k-1}Y_{1,r-1}(1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\
&\quad \left. \times (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,1}^{-1}) \right) \\
&= a^{\Lambda_{k-1} + \Lambda_r} \sum_{b \in B(\Lambda_k + \Lambda_r)_{s_{k+1} s_{k+2} \cdots s_r}} C(b) \mu(b) \\
&\quad + a^{\Lambda_{k-1} + \Lambda_r} \sum_{b \in B(\Lambda_{k-1} + \Lambda_{r-1})_{s_1 \cdots s_{k-1} s_{k+1} s_{k+2} \cdots s_{r-1}}} \mu'(b), \tag{6.15}
\end{aligned}$$

where μ is the monomial realization of $B(\Lambda_k + \Lambda_r)$ such that the highest weight vector is realized by $Y_{2,k}Y_{1,r} \in \mathcal{Y}$, μ' is the monomial realization of $B(\Lambda_{k-1} + \Lambda_{r-1})$ such that the highest weight vector is realized by $Y_{1,k-1}Y_{1,r-1} \in \mathcal{Y}$, and $C(b)$ are some positive integers. We also obtain

$$\begin{aligned}
& (\varphi_{\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{v}))_r(a; \mathbf{Y}) \\
&= a^{\Lambda_r + 2\Lambda_{k-1}} Y_{1,r}Y_{2,k}^2(1 + A_{1,r}^{-1}(1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1})^2) \tag{6.16} \\
&= a^{2\Lambda_{k-1} + \Lambda_r} \sum_{b \in B(2\Lambda_k + \Lambda_r)_{s_{k+1} s_{k+2} \cdots s_r}} C'(b) \mu''(b), \tag{6.17}
\end{aligned}$$

where μ'' is the monomial realization of $B(2\Lambda_k + \Lambda_r)$ such that the highest weight vector is realized by $Y_{2,k}^2Y_{1,r} \in \mathcal{Y}$, and $C'(b)$ are some positive integers.

(iv) For j and k with $1 \leq j < k \leq r-2$,

$$\begin{aligned}
& (\varphi_{\mu_j \cdots \mu_{k-2} \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{v}))_j(a; \mathbf{Y}) = a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_r} (Y_{2,k}Y_{2,j}Y_{1,r} \\
&+ \frac{Y_{2,k}Y_{2,j}Y_{1,r-1}^2}{Y_{2,r}}(1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1}A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \tag{6.18} \\
&\quad \times (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1}A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,j+1}^{-1}))
\end{aligned}$$

$$\begin{aligned}
&= a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_r} \sum_{b \in B(\Lambda_j + \Lambda_k + 2\Lambda_r)_{s_{k+1} s_{k+2} \cdots s_r}} C(b) \mu(b) \tag{6.19} \\
&\quad + a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_r} \sum_{b \in B(\Lambda_j + \Lambda_{k-1} + \Lambda_{r-1})_{s_{j+1} \cdots s_{k-1} s_{k+1} s_{k+2} \cdots s_{r-1}}} \mu'(b),
\end{aligned}$$

where μ is the monomial realization of $B(\Lambda_j + \Lambda_k + \Lambda_r)$ such that the highest weight vector is realized by $Y_{2,j}Y_{2,k}Y_{1,r} \in \mathcal{Y}$, μ' is the monomial realization of $B(\Lambda_j + \Lambda_{k-1} + \Lambda_{r-1})$ such that the highest weight vector is realized by $Y_{2,j}Y_{1,k-1}Y_{1,r-1} \in \mathcal{Y}$, and $C(b)$ are some positive integers.

Theorem 6.5. Let $G = \mathrm{SO}_{2r}(\mathbb{C})$ be the classical algebraic group of type D_r .

(i) For $k \in [1, r]$, we obtain

$$(\varphi_V^G)_k(a; \mathbf{Y}) = a^{\Lambda_k} \sum_{b \in B(\Lambda_k)_{c^2 > 2r-k}} C(b) \mu(b),$$

where $\mu : B(\Lambda_k) \rightarrow \mathcal{Y}$ is the monomial realization of $B(\Lambda_k)$ such that the highest weight vector in $B(\Lambda_k)$ is realized by $Y_{1,k} \in \mathcal{Y}$, and $C(b)$ are some positive integers. Furthermore, we have

$$(\varphi_{\mu_{r-1}(\mathbf{V})}^G)_{r-1}(a; \mathbf{Y}) = a^{\Lambda_{r-2}} Y_{2,r-1} = a^{\Lambda_{r-2}} \sum_{b \in B(\Lambda_{r-1})_e} \mu'_{r-1}(b),$$

$$(\varphi_{\mu_r \mu_{r-1}(\mathbf{V})}^G)_r(a; \mathbf{Y}) = a^{\Lambda_{r-2}} Y_{2,r} = a^{\Lambda_{r-2}} \sum_{b \in B(\Lambda_r)_e} \mu'_r(b).$$

For $k \in [1, r-2]$,

$$(\varphi_{\mu_k \mu_{k+1} \cdots \mu_{r-3} \mu_{r-2} \mu_r \mu_{r-1}(\mathbf{V})}^G)_k(a; \mathbf{Y}) = a^{\Lambda_{k-1} + \Lambda_{r-2}} Y_{2,k} = a^{\Lambda_{k-1} + \Lambda_{r-2}} \sum_{b \in B(\Lambda_k)_e} \mu'_k(b),$$

where μ'_s ($1 \leq s \leq r$) is the monomial realization of $B(\Lambda_s)$ such that the highest weight vector is realized by $Y_{2,s} \in \mathcal{Y}$.

(ii) For k and l with $1 \leq k \leq l \leq r-3$,

$$\begin{aligned} & (\varphi_{\mu_l \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_l(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1} + \Lambda_{l+1}} Y_{2,k} Y_{1,l+1} (1 + A_{1,l+1}^{-1} + A_{1,l+1}^{-1} A_{1,l}^{-1} + \cdots + A_{1,l+1}^{-1} A_{1,l}^{-1} \cdots A_{1,k+1}^{-1}) \end{aligned} \quad (6.20)$$

$$= a^{\Lambda_{k-1} + \Lambda_{l+1}} \sum_{b \in B(\Lambda_k + \Lambda_{l+1})_{s_{k+1} s_{k+2} \cdots s_{l+1}}} \mu(b), \quad (6.21)$$

where μ is the monomial realization of $B(\Lambda_k + \Lambda_{l+1})$ such that the highest weight vector in $B(\Lambda_k + \Lambda_{l+1})$ is realized by $Y_{2,k} Y_{1,l+1} \in \mathcal{Y}$.

(iii) For $k \in [1, r-2]$, we obtain

$$\begin{aligned} & (\varphi_{\mu_{r-2} \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_{r-2}(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{2,k} Y_{1,r-1} Y_{1,r} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\ & \quad \times (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \\ & \quad + a^{\Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{1,k-1} Y_{1,r-2} (1 + A_{1,r-2}^{-1} + \cdots + A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}) \end{aligned} \quad (6.22)$$

$$\begin{aligned} & \times (1 + A_{1,k-1}^{-1} + A_{1,k-1}^{-1} A_{1,k-2}^{-1} + \cdots + A_{1,k-1}^{-1} A_{1,k-2}^{-1} \cdots A_{1,1}^{-1}) \\ &= a^{\Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} \sum_{b \in B(\Lambda_k + \Lambda_{r-1} + \Lambda_r)_{s_{k+1} \cdots s_{r-1} s_r}} C(b) \mu(b) \quad (6.23) \\ & \quad + a^{\Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} \sum_{b \in B(\Lambda_{k-1} + \Lambda_{r-2})_{s_1 \cdots s_{k-1} s_{k+1} \cdots s_{r-3} s_{r-2}}} \mu'(b), \end{aligned}$$

where μ is the monomial realization of $B(\Lambda_k + \Lambda_{r-1} + \Lambda_r)$ such that the highest weight vector is realized by $Y_{2,k} Y_{1,r-1} Y_{1,r} \in \mathcal{Y}$ and $C(b)$ are some integers, μ' is the monomial realization of $B(\Lambda_{k-1} + \Lambda_{r-2})$ such that the highest weight vector is realized by $Y_{1,k-1} Y_{1,r-2} \in \mathcal{Y}$.

(iv) For $k \in [1, r-2]$,

$$\begin{aligned} & (\varphi_{\mu_{r-1}\mu_{r-2}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{r-1}(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1}+\Lambda_r} Y_{2,k} Y_{1,r} (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}), \end{aligned} \quad (6.24)$$

$$= a^{\Lambda_{k-1}+\Lambda_r} \sum_{b \in B(\Lambda_k+\Lambda_r)_{s_{k+1}\cdots s_{r-1} s_r}} \mu(b), \quad (6.25)$$

where μ is the monomial realization of $B(\Lambda_k + \Lambda_r)$ such that the highest weight vector is realized by $Y_{2,k} Y_{1,r} \in \mathcal{Y}$.

$$\begin{aligned} & (\varphi_{\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_r(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1}+\Lambda_{r-1}} Y_{2,k} Y_{1,r-1} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}) \end{aligned} \quad (6.26)$$

$$= a^{\Lambda_{k-1}+\Lambda_{r-1}} \sum_{b \in B(\Lambda_k+\Lambda_{r-1})_{s_{k+1} s_{k+2} \cdots s_{r-1}}} \mu'(b), \quad (6.27)$$

where μ' is the monomial realization of $B(\Lambda_k + \Lambda_{r-1})$ such that the highest weight vector is realized by $Y_{2,k} Y_{1,r-1} \in \mathcal{Y}$.

(v) For j and k with $1 \leq j < k \leq r-2$,

$$\begin{aligned} & (\varphi_{\mu_j\cdots\mu_{k-2}\mu_{k-1}\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_j(a; \mathbf{Y}) \\ &= a^{\Lambda_{j-1}+\Lambda_{k-1}+\Lambda_{r-1}+\Lambda_r} Y_{2,j} Y_{2,k} Y_{1,r-1} Y_{1,r} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}) \\ & \quad \times (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \\ &+ a^{\Lambda_{j-1}+\Lambda_{k-1}+\Lambda_{r-1}+\Lambda_r} Y_{2,j} Y_{1,k-1} Y_{1,r-2} (1 + A_{1,r-2}^{-1} + \cdots + A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \\ & \quad \times (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} A_{1,k-2}^{-1} \cdots A_{1,j+1}^{-1}) \\ &= a^{\Lambda_{j-1}+\Lambda_{k-1}+\Lambda_{r-1}+\Lambda_r} \sum_{b \in B(\Lambda_j+\Lambda_k+\Lambda_{r-1}+\Lambda_r)_{s_{k+1}\cdots s_{r-1} s_r}} C(b) \mu(b) \\ &+ a^{\Lambda_{j-1}+\Lambda_{k-1}+\Lambda_{r-1}+\Lambda_r} \sum_{B(\Lambda_j+\Lambda_{k-1}+\Lambda_{r-2})_{s_{j+1}\cdots s_{k-2} s_{k-1} s_{k+1}\cdots s_{r-3} s_{r-2}}} \mu'(b), \end{aligned} \quad (6.29)$$

where μ is the monomial realization of $B(\Lambda_j + \Lambda_k + \Lambda_{r-1} + \Lambda_r)$ such that the highest weight vector is realized by $Y_{2,j} Y_{2,k} Y_{1,r-1} Y_{1,r} \in \mathcal{Y}$, and μ' is the monomial realization of $B(\Lambda_j + \Lambda_{k-1} + \Lambda_{r-2})$ such that the highest weight vector is mapped to $Y_{2,j} Y_{1,k-1} Y_{1,r-2} \in \mathcal{Y}$, and $C(b)$ are some positive integers.

7 The proof of main theorem

In this section, we prove Theorem 6.3, 6.4 and 6.5. Let $\Sigma_0 := (\mathbf{V}, \tilde{B}(\mathbf{i}))$ be the initial seed of $\mathbb{C}[G^{e,c^2}]$.

7.1 The proof of Theorem 6.3

First, we shall prove the case $G = \mathrm{SO}_{2r+1}(\mathbb{C})$. We start by setting the Laurent monomials as follows:

$$B(l, k) := \begin{cases} \frac{Y_{l,k}}{Y_{l,k-1}} & \text{if } 1 \leq k \leq r-1, \\ \frac{Y_{l,r}^2}{Y_{l,r-1}} & \text{if } k = r, \\ \frac{Y_{l,r}}{Y_{l+1,r}} & \text{if } k = 0, \\ \frac{Y_{l,r-1}}{Y_{l+1,r}} & \text{if } k = \bar{r}, \\ \frac{Y_{l,|k|-1}}{Y_{l+1,|k|}} & \text{if } \overline{r-1} \leq k \leq \bar{1}, \end{cases} \quad (7.1)$$

where for $1 \leq l \leq r$, we set $|l| = |\bar{l}| = l$.

Proposition 7.1. (i) For $k \in [1, r]$, the initial cluster variables $(\varphi_V^G)_k(a; \mathbf{Y})$ in $\mathbb{C}[G^{e,c^2}]$ are described as

$$(\varphi_V^G)_k(a; \mathbf{Y}) = a^{\Lambda_k} \sum_{b \in B(\Lambda_k)_{c^2}_{>2r-k}} \mu(b),$$

where $\mu : B(\Lambda_k) \rightarrow \mathcal{Y}$ is the monomial realization of $B(\Lambda_k)$ in Theorem 6.3 (i).

(ii) For $k \in [1, r]$, the frozen cluster variables $(\varphi_V^G)_{-k}(a; \mathbf{Y})$, $(\varphi_V^G)_{r+k}(a; \mathbf{Y})$ in $\mathbb{C}[G^{e,c^2}]$ are described as

$$(\varphi_V^G)_{-k}(a; \mathbf{Y}) = a^{\Lambda_k} Y_{1,k} Y_{2,k}, \quad (\varphi_V^G)_{r+k}(a; \mathbf{Y}) = a^{\Lambda_k}.$$

[Proof.]

First, let $k \in [1, r-1]$, and recall the fundamental representation $V(\Lambda_k)$ of type B_r in 2.3. By (2.10), (2.11) and (3.2), for $i \in [1, r-1]$, $j \in [1, r]$, we get

$$\overline{s_i} v_j = \begin{cases} v_{i+1} & \text{if } j = i, \\ -v_i & \text{if } j = i+1, \\ v_j & \text{if otherwise,} \end{cases} \quad \overline{s_r} v_j = \begin{cases} v_{\bar{r}} & \text{if } j = r, \\ v_j & \text{if } 1 \leq j < r. \end{cases}$$

Taking into account these formulas, we obtain

$$\overline{c_{>2r-k}^2} v_1 \wedge \cdots \wedge v_k = \overline{s_1} \cdots \overline{s_k} (v_1 \wedge \cdots \wedge v_k) = v_2 \wedge \cdots \wedge v_{k+1}. \quad (7.2)$$

Just as in Example 6.2, the value of $(\varphi_V^G)_k(a; \mathbf{Y})$ coincides with

$$a^{\Lambda_k} \langle v_2 \wedge \cdots \wedge v_{k+1}, \alpha_1^\vee(Y_{2,1}) y_1(Y_{2,1}) \cdots \alpha_r^\vee(Y_{2,r}) y_r(Y_{2,r}) \alpha_1^\vee(Y_{1,1}) y_1(Y_{1,1}) \cdots \alpha_k^\vee(Y_{1,k}) y_k(Y_{1,k}) v_1 \wedge \cdots \wedge v_k \rangle.$$

Using (2.8), (2.10) and (2.11) repeatedly, for $i \in [1, k]$, one obtain

$$\alpha_1^\vee(Y_{1,1}) y_1(Y_{1,1}) \cdots \alpha_k^\vee(Y_{1,k}) y_k(Y_{1,k}) v_i = \frac{Y_{1,i}}{Y_{1,i-1}} v_i + v_{i+1},$$

which means that $\alpha_1^\vee(Y_{1,1}) y_1(Y_{1,1}) \cdots \alpha_k^\vee(Y_{1,k}) y_k(Y_{1,k}) v_1 \wedge \cdots \wedge v_k$ is a linear combination of $v_1 \wedge \cdots \wedge v_s \wedge v_{s+2} \wedge \cdots \wedge v_{k+1}$ ($0 \leq s \leq k$) with the coefficient

$B(1,1)B(1,2)\cdots B(1,s)$. Similarly, we can also verify that for s ($0 \leq s \leq k$) the coefficient of $v_2 \wedge \cdots \wedge v_{k+1}$ in $\alpha_1^\vee(Y_{2,1})y_1(Y_{2,1})\cdots\alpha_r^\vee(Y_{2,r})y_r(Y_{2,r})(v_1 \wedge \cdots \wedge v_s \wedge v_{s+2} \wedge \cdots \wedge v_{k+1})$ is $B(2,s+2)B(2,s+3)\cdots B(2,k+1)$. Hence, we get

$$(\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y}) = a^{\Lambda_k} \sum_{0 \leq s \leq k} B(1,1)B(1,2)\cdots B(1,s)B(2,s+2)\cdots B(2,k+1). \quad (7.3)$$

The definition of the monomial realization implies that

$$\begin{aligned} & \tilde{f}_s B(1,1)\cdots B(1,s-1)B(1,s)B(2,s+2)\cdots B(2,k+1) \\ &= B(1,1)\cdots B(1,s-1)B(1,s)B(2,s+2)\cdots B(2,k+1)A_{1,s}^{-1} \\ &= B(1,1)\cdots B(1,s-1)B(2,s+1)B(2,s+2)\cdots B(2,k+1), \end{aligned}$$

where we use $B(1,s)A_{1,s}^{-1} = \frac{Y_{1,s}}{Y_{1,s-1}} \frac{Y_{1,s-1}Y_{2,s+1}}{Y_{1,s}Y_{2,s}} = B(2,s+1)$. Therefore, the conclusion

$$\begin{aligned} (\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y}) &= a^{\Lambda_k} Y_{1,k} (1 + A_{1,k}^{-1} + A_{1,k}^{-1}A_{1,k-1}^{-1} + \cdots + A_{1,k}^{-1}A_{1,k-1}^{-1}\cdots A_{1,1}^{-1}) \\ &= a^{\Lambda_k} \sum_{b \in B(\Lambda_k)_{c_{>2r-k}^2}} \mu(b) \end{aligned}$$

follows from Theorem 5.4 and the easy fact $B(1,1)B(1,2)\cdots B(1,k) = Y_{1,k}$. By the same argument, the frozen cluster variable $(\varphi_{\mathbf{V}}^G)_{-k}(a; \mathbf{Y}) = \Delta_{\Lambda_k, \Lambda_k} \circ \bar{x}^G(a; \mathbf{Y})$ coincides with the coefficient of $v_1 \wedge \cdots \wedge v_k$ in

$$a\alpha_1^\vee(Y_{2,1})y_1(Y_{2,1})\cdots\alpha_r^\vee(Y_{2,r})y_r(Y_{2,r})\alpha_1^\vee(Y_{1,1})y_1(Y_{1,1})\cdots\alpha_k^\vee(Y_{1,k})y_k(Y_{1,k})v_1 \wedge \cdots \wedge v_k, \quad (7.4)$$

which equals to $a^{\Lambda_k} B(1,1)B(1,2)\cdots B(1,k)B(2,1)B(2,2)\cdots B(2,k) = a^{\Lambda_k} Y_{1,k}Y_{2,k}$. The frozen cluster variable $(\varphi_{\mathbf{V}}^G)_{r+k}(a; \mathbf{Y}) = \Delta_{\Lambda_k, c^{-2}\Lambda_k} \circ \bar{x}^G(a; \mathbf{Y})$ coincides with the coefficient of

$$\begin{cases} v_3 \wedge \cdots \wedge v_{k+2} & \text{if } k < r-1, \\ v_3 \wedge \cdots \wedge v_r \wedge v_{\bar{1}} & \text{if } k = r-1, \end{cases}$$

in (7.4). It is equal to a^{Λ_k} .

Next, we now turn to the case $k = r$. We need to recall the spin representation in 2.3. From (2.13), (2.14) and (3.2), we see that

$$\overline{c_{>r}^2}(+, +, +, \cdots, +) = \overline{s_1} \cdots \overline{s_r}(+, +, +, \cdots, +) = (-, +, +, \cdots, +).$$

Just as in the case $k < r$, the value of $(\varphi_{\mathbf{V}}^G)_r(a; \mathbf{Y})$ coincides with

$$\begin{aligned} & a^{\Lambda_r} \langle (-, +, +, \cdots, +), \alpha_1^\vee(Y_{2,1})y_1(Y_{2,1})\cdots\alpha_r^\vee(Y_{2,r})y_r(Y_{2,r}) \\ & \quad \alpha_1^\vee(Y_{1,1})y_1(Y_{1,1})\cdots\alpha_r^\vee(Y_{1,r})y_r(Y_{1,r})(+, +, +, \cdots, +) \rangle. \end{aligned}$$

From (2.12) and (2.13), we see that $\alpha_1^\vee(Y_{1,1})y_1(Y_{1,1})\cdots\alpha_r^\vee(Y_{1,r})y_r(Y_{1,r})(+, +, +, \cdots, +)$ is a linear combination of $(+, +, \cdots, +)$ and $(+, \cdots, +, \overset{i}{-}, \overset{i+1}{+}, \cdots, +)$ ($1 \leq i \leq r$) whose coefficients are $Y_{1,r}$ and $Y_{1,i-1}$, respectively. Similarly, the coefficient of $(-, +, \cdots, +, +)$ in $\alpha_1^\vee(Y_{2,1})y_1(Y_{2,1})\cdots\alpha_r^\vee(Y_{2,r})y_r(Y_{2,r})(+, +, \cdots, +, +)$ is 1, the one in $\alpha_1^\vee(Y_{2,1})y_1(Y_{2,1})\cdots\alpha_r^\vee(Y_{2,r})y_r(Y_{2,r})(+, \cdots, +, \overset{i}{-}, \overset{i+1}{+}, \cdots, +)$ is $\frac{Y_{2,r}}{Y_{2,i}}$

if $i < r$, and is $\frac{1}{Y_{2,r}}$ if $i = r$. Hence, it follows that

$$\begin{aligned} (\varphi_{\mathbf{V}}^G)_r(a; \mathbf{Y}) &= a^{\Lambda_r} \left(Y_{1,r} + \frac{Y_{1,r-1}}{Y_{2,r}} + \sum_{i=1}^{r-1} \frac{Y_{1,i-1}Y_{2,r}}{Y_{2,i}} \right) \\ &= a^{\Lambda_r} \sum_{b \in B(\Lambda_r)_{c_{>r}^2}} \mu(b). \end{aligned} \quad (7.5)$$

By the same argument, we can prove that $(\varphi_{\mathbf{V}}^G)_{-r}(a; \mathbf{Y}) = a^{\Lambda_r} Y_{1,r} Y_{2,r}$ and $(\varphi_{\mathbf{V}}^G)_{2r}(a; \mathbf{Y}) = a^{\Lambda_r}$.

In the proof, we found the explicit form (7.3) of $(\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y})$ for $k \in [1, r-1]$, which can be rewritten as

$$\begin{aligned} (\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y}) &= a^{\Lambda_k} \sum_{0 \leq s \leq k} B(1,1)B(1,2) \cdots B(1,s)B(2,s+2) \cdots B(2,k+1) \\ &= a^{\Lambda_k} (B(1,1) \cdots B(1,k) + \sum_{0 \leq s \leq k-1} B(1,1) \cdots B(1,s)B(2,s+2) \cdots B(2,k+1)) \\ &= a^{\Lambda_k} (Y_{1,k} + B(2,k+1) \sum_{0 \leq s \leq k-1} B(1,1) \cdots B(1,s)B(2,s+2) \cdots B(2,k)) \\ &= a^{\Lambda_k} (Y_{1,k} + B(2,k+1)(a^{-\Lambda_{k-1}}(\varphi_{\mathbf{V}}^G)_{k-1}(a; \mathbf{Y}))). \end{aligned}$$

Thus, we obtain the relation between two cluster variables:

$$a^{-\Lambda_k}(\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y}) = Y_{1,k} + B(2,k+1)(a^{-\Lambda_{k-1}}(\varphi_{\mathbf{V}}^G)_{k-1}(a; \mathbf{Y})). \quad (7.6)$$

Lemma 7.2. *For $k \in [1, r]$, the cluster variables $(\varphi_{\mu_k \mu_{k+1} \cdots \mu_r(\mathbf{V})}^G)_k(a; \mathbf{Y})$ in $\mathbb{C}[G^{e,c^2}]$ are described as*

$$(\varphi_{\mu_k \mu_{k+1} \cdots \mu_r(\mathbf{V})}^G)_k(a; \mathbf{Y}) = \begin{cases} a^{\Lambda_{k-1} + \Lambda_{r-1}} Y_{2,k} & \text{if } k < r, \\ a^{\Lambda_{r-1}} Y_{2,r} & \text{if } k = r. \end{cases}$$

[Proof.]

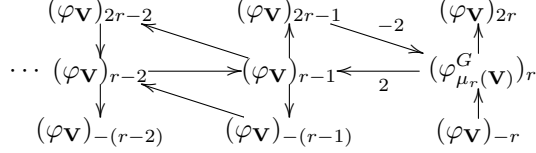
We prove this statement by induction on $(r-k)$. If $r-k=0$, so that $k=r$, then the mutation diagram (6.3) implies that

$$(\varphi_{\mu_r(\mathbf{V})}^G)_r(a; \mathbf{Y}) = \frac{(\varphi_{\mathbf{V}}^G)_{r-1}(a; \mathbf{Y})(\varphi_{\mathbf{V}}^G)_{2r}(a; \mathbf{Y}) + (\varphi_{\mathbf{V}}^G)_{2r-1}(a; \mathbf{Y})(\varphi_{\mathbf{V}}^G)_{-r}(a; \mathbf{Y})}{(\varphi_{\mathbf{V}}^G)_r(a; \mathbf{Y})}.$$

Applying Proposition 7.1 (ii), (6.5) for $k=r-1$ and (7.5), we obtain

$$\begin{aligned} &(\varphi_{\mu_r(\mathbf{V})}^G)_r(a; \mathbf{Y}) \\ &= \frac{a^{\Lambda_{r-1} + \Lambda_r} Y_{1,r-1} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,1}^{-1}) + Y_{1,r} Y_{2,r}}{a^{\Lambda_r} Y_{1,r} + \frac{Y_{1,r-1}}{Y_{2,r}} + \sum_{i=1}^{r-1} \frac{Y_{1,i-1} Y_{2,r}}{Y_{2,i}}} \\ &= a^{\Lambda_{r-1}} \frac{Y_{1,r} Y_{2,r} + Y_{1,r-1} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,1}^{-1})}{Y_{1,r} + \frac{Y_{1,r-1}}{Y_{2,r}} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,1}^{-1})} \\ &= a^{\Lambda_{r-1}} Y_{2,r}. \end{aligned}$$

Next, we consider the the cluster variable $(\varphi_{\mu_{r-1}\mu_r(\mathbf{V})}^G)_{r-1}(a; \mathbf{Y})$. In the rest of this proof, we abbreviate $(\varphi_{\mathbf{T}}^G)_s(a; \mathbf{Y})$ to $(\varphi_{\mathbf{T}}^G)_s$ for $s \in [1, r]$ and clusters \mathbf{T} . By Lemma 4.11, the mutation diagram of $\mu_r(\Sigma_0)$ is as follows:

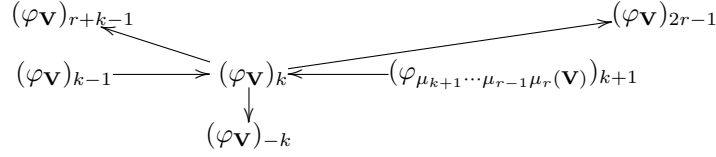


Therefore, we get

$$\begin{aligned}
(\varphi_{\mu_{r-1}\mu_r(\mathbf{V})}^G)_{r-1} &= \frac{(\varphi_{\mathbf{V}}^G)_{r-2}(\varphi_{\mu_r(\mathbf{V})}^G)_r^2 + (\varphi_{\mathbf{V}}^G)_{2r-2}(\varphi_{\mathbf{V}}^G)_{2r-1}(\varphi_{\mathbf{V}}^G)_{-(r-1)}}{(\varphi_{\mathbf{V}}^G)_{r-1}} \\
&= \frac{(a^{\Lambda_{r-2}-\Lambda_{r-1}} \frac{Y_{2,r-1}}{Y_{2,r}^2} (\varphi_{\mathbf{V}}^G)_{r-1} - a^{\Lambda_{r-2}} \frac{Y_{1,r-1}Y_{2,r-1}}{Y_{2,r}^2}) a^{2\Lambda_{r-1}} Y_{2,r}^2 + a^{\Lambda_{r-2}+2\Lambda_{r-1}} Y_{1,r-1} Y_{2,r-1}}{(\varphi_{\mathbf{V}}^G)_{r-1}} \\
&= a^{\Lambda_{r-2}+\Lambda_{r-1}} Y_{2,r-1},
\end{aligned}$$

where we use Proposition 7.1 (ii) and (7.6) in the second equality.

Next, we assume that $r-k > 1$. It follows from Lemma 4.11 that the arrows incident to the vertex $(\varphi_{\mathbf{V}})_k$ in the mutation diagram of $\mu_{k+1} \cdots \mu_{r-1}\mu_r(\Sigma_0)$ are as follows:



Thus, we obtain

$$\begin{aligned}
(\varphi_{\mu_k\mu_{k+1}\cdots\mu_{r-1}\mu_r(\mathbf{V})}^G)_k &= \frac{(\varphi_{\mathbf{V}}^G)_{k-1}(\varphi_{\mu_{k+1}\cdots\mu_{r-1}\mu_r(\mathbf{V})}^G)_{k+1} + (\varphi_{\mathbf{V}}^G)_{r+k-1}(\varphi_{\mathbf{V}}^G)_{2r-1}(\varphi_{\mathbf{V}}^G)_{-k}}{(\varphi_{\mathbf{V}}^G)_k} \\
&= \frac{(a^{\Lambda_{k-1}-\Lambda_k} \frac{Y_{2,k}}{Y_{2,k+1}} (\varphi_{\mathbf{V}}^G)_k - a^{\Lambda_{k-1}} \frac{Y_{1,k}Y_{2,k}}{Y_{2,k+1}}) a^{\Lambda_k+\Lambda_{r-1}} Y_{2,k+1} + a^{\Lambda_k+\Lambda_{k-1}+\Lambda_{r-1}} Y_{1,k} Y_{2,k}}{(\varphi_{\mathbf{V}}^G)_k} \\
&= a^{\Lambda_{k-1}+\Lambda_{r-1}} Y_{2,k},
\end{aligned}$$

where we use Proposition 7.1 (ii), (7.6) and the induction hypothesis in the second equality. \square

[Proof of Theorem 6.3 (ii)]

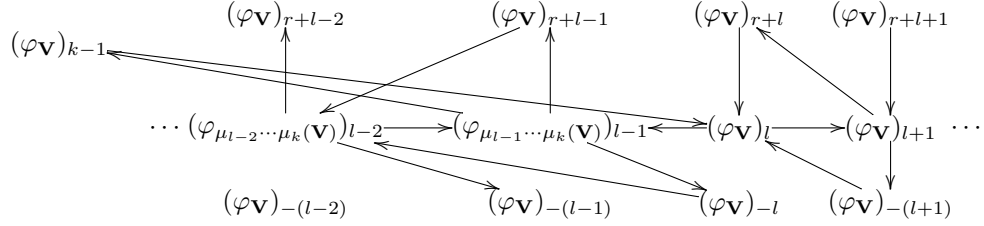
For fixed k , we use the induction on $(l-k)$ to prove (6.6). If $l-k = 0$, so

that $k = l$, then the mutation diagram (6.2) means that

$$\begin{aligned}
(\varphi_{\mu_k(\mathbf{V})}^G)_k &= \frac{(\varphi_{\mathbf{V}}^G)_{k-1}(\varphi_{\mathbf{V}}^G)_{r+k}(\varphi_{\mathbf{V}}^G)_{-(k+1)} + (\varphi_{\mathbf{V}}^G)_{k+1}(\varphi_{\mathbf{V}}^G)_{r+k-1}(\varphi_{\mathbf{V}}^G)_{-k}}{(\varphi_{\mathbf{V}}^G)_k} \\
&= \frac{(\varphi_{\mathbf{V}}^G)_{k-1}a^{\Lambda_k+\Lambda_{k+1}}Y_{1,k+1}Y_{2,k+1} + (\varphi_{\mathbf{V}}^G)_{k+1}a^{\Lambda_{k-1}+\Lambda_k}Y_{1,k}Y_{2,k}}{(\varphi_{\mathbf{V}}^G)_k} \\
&= \frac{(\varphi_{\mathbf{V}}^G)_k a^{\Lambda_{k-1}+\Lambda_{k+1}}Y_{2,k}Y_{1,k+1} + (\varphi_{\mathbf{V}}^G)_k a^{\Lambda_{k-1}+\Lambda_{k+1}} \frac{Y_{1,k}Y_{2,k}Y_{2,k+2}}{Y_{2,k+1}}}{(\varphi_{\mathbf{V}}^G)_k} \\
&= a^{\Lambda_{k-1}+\Lambda_{k+1}} \left(Y_{2,k}Y_{1,k+1} + \frac{Y_{1,k}Y_{2,k}Y_{2,k+2}}{Y_{2,k+1}} \right),
\end{aligned}$$

where we use Proposition 7.1 (ii) in the second equality, and (7.6) for k and $k+1$ in the third equality.

Next, we assume that $l - k > 0$. The vertices and arrows around the vertex $(\varphi_{\mathbf{V}})_l$ in the mutation diagram of $\mu_{l-1} \cdots \mu_k(\Sigma_0)$ ($l \leq r-2$) are as follows (Lemma 4.11):



It follows from this diagram that

$$(\varphi_{\mu_{l-1} \cdots \mu_k(\mathbf{V})}^G)_l = \frac{(\varphi_{\mathbf{V}}^G)_{l+1}(\varphi_{\mu_{l-1} \cdots \mu_k(\mathbf{V})}^G)_{l-1} + (\varphi_{\mathbf{V}}^G)_{k-1}(\varphi_{\mathbf{V}}^G)_{r+l}(\varphi_{\mathbf{V}}^G)_{-(l+1)}}{(\varphi_{\mathbf{V}}^G)_l}. \quad (7.7)$$

Using the induction hypothesis, the cluster variable $(\varphi_{\mu_{l-1} \cdots \mu_k(\mathbf{V})}^G)_{l-1}$ is

$$a^{\Lambda_{k-1}+\Lambda_l}Y_{2,k}Y_{1,l}(1 + A_{1,l}^{-1} + A_{1,l}^{-1}A_{1,l-1}^{-1} + \cdots + A_{1,l}^{-1}A_{1,l-1}^{-1} \cdots A_{1,k+1}^{-1}).$$

In conjunction with (6.5), it follows that $(\varphi_{\mu_{l-1} \cdots \mu_k(\mathbf{V})}^G)_{l-1}$ coincides with

$$\begin{aligned}
&a^{\Lambda_{k-1}+\Lambda_l}Y_{2,k}Y_{1,l}(a^{-\Lambda_l}Y_{1,l}^{-1}(\varphi_{\mathbf{V}}^G)_l - a^{-\Lambda_{k-1}}Y_{1,k-1}^{-1}(\varphi_{\mathbf{V}}^G)_{k-1}A_{1,l}^{-1}A_{1,l-1}^{-1} \cdots A_{1,k}^{-1}) \\
&= a^{\Lambda_{k-1}}Y_{2,k}(\varphi_{\mathbf{V}}^G)_l - a^{\Lambda_l}Y_{2,l+1}(\varphi_{\mathbf{V}}^G)_{k-1}. \quad (7.8)
\end{aligned}$$

Applying Proposition 7.1 (ii), (7.6) and this formula to (7.7), we see that

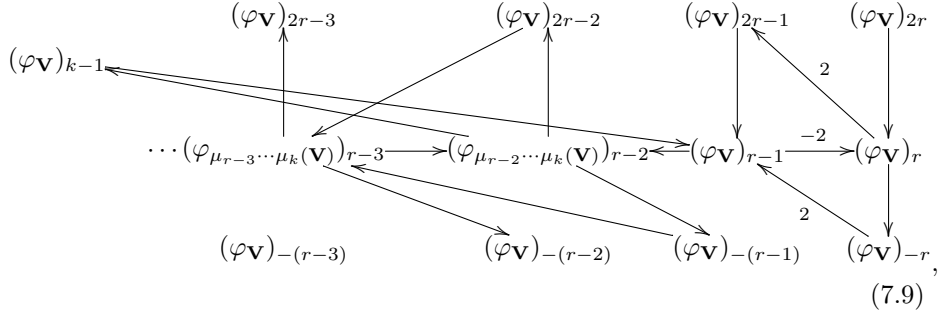
$$\begin{aligned}
& (\varphi_{\mu_l \mu_{l-1} \dots \mu_k(\mathbf{V})}^G)_l \\
&= \frac{(a^{\Lambda_{l+1}} Y_{1,l+1} + a^{\Lambda_{l+1} - \Lambda_l} \frac{Y_{2,l+2}}{Y_{2,l+1}} (\varphi_{\mathbf{V}}^G)_l) (a^{\Lambda_{k-1}} Y_{2,k} (\varphi_{\mathbf{V}}^G)_l - a^{\Lambda_l} Y_{2,l+1} (\varphi_{\mathbf{V}}^G)_{k-1})}{(\varphi_{\mathbf{V}}^G)_l} \\
&\quad + \frac{(\varphi_{\mathbf{V}}^G)_{k-1} a^{\Lambda_l + \Lambda_{l+1}} Y_{1,l+1} Y_{2,l+1}}{(\varphi_{\mathbf{V}}^G)_l} \\
&= a^{\Lambda_{l+1} - \Lambda_l + \Lambda_{k-1}} \frac{Y_{2,l+2} Y_{2,k}}{Y_{2,l+1}} (\varphi_{\mathbf{V}}^G)_l + a^{\Lambda_{l+1} + \Lambda_{k-1}} Y_{1,l+1} Y_{2,k} - a^{\Lambda_{l+1}} Y_{2,l+2} (\varphi_{\mathbf{V}}^G)_{k-1} \\
&= a^{\Lambda_{l+1} + \Lambda_{k-1}} Y_{1,l+1} Y_{2,k} (1 + A_{1,l+1}^{-1} + A_{1,l+1}^{-1} A_{1,l}^{-1} + \dots + A_{1,l+1}^{-1} A_{1,l}^{-1} \dots A_{1,1}^{-1}) \\
&\quad - a^{\Lambda_{l+1} + \Lambda_{k-1}} Y_{1,k-1} Y_{2,l+2} (1 + A_{1,k-1}^{-1} + A_{1,k-1}^{-1} A_{1,k-2}^{-1} + \dots + A_{1,k-1}^{-1} A_{1,k-2}^{-1} \dots A_{1,1}^{-1}) \\
&= a^{\Lambda_{l+1} + \Lambda_{k-1}} Y_{1,l+1} Y_{2,k} (1 + A_{1,l+1}^{-1} + A_{1,l+1}^{-1} A_{1,l}^{-1} + \dots + A_{1,l+1}^{-1} A_{1,l}^{-1} \dots A_{1,k+1}^{-1}),
\end{aligned}$$

where we use (6.5) in the third equality, and $Y_{1,l+1} Y_{2,k} A_{1,l+1}^{-1} A_{1,l}^{-1} \dots A_{1,1}^{-1} = Y_{1,k-1} Y_{2,l+2}$ in the fourth equality. Hence, we get (6.6) for all l ($k \leq l \leq r-2$).

By the same argument as in the proof of Proposition 7.1, the equation (6.7) follows from (6.6). \square

[Proof of Theorem 6.3 (iii)]

Using Lemma 4.11 repeatedly, we see that the mutation diagram of $\mu_{r-2} \dots \mu_{k+1} \mu_k(\Sigma_0)$ is as follows:



which implies that

$$(\varphi_{\mu_{r-1} \mu_{r-2} \dots \mu_k(\mathbf{V})}^G)_{r-1} = \frac{(\varphi_{\mathbf{V}}^G)_r^2 (\varphi_{\mu_{r-2} \dots \mu_k(\mathbf{V})}^G)_{r-2} + (\varphi_{\mathbf{V}}^G)_{k-1} (\varphi_{\mathbf{V}}^G)_{2r-1} (\varphi_{\mathbf{V}}^G)_{-r}^2}{(\varphi_{\mathbf{V}}^G)_{r-1}}. \quad (7.10)$$

On the other hand, from (6.5) and (7.5), we have

$$(\varphi_{\mathbf{V}}^G)_r = a^{\Lambda_r} \left(Y_{1,r} + \frac{a^{-\Lambda_{r-1}}}{Y_{2,r}} (\varphi_{\mathbf{V}}^G)_{r-1} \right). \quad (7.11)$$

Just as in (7.8), we see that

$$(\varphi_{\mu_{r-2} \dots \mu_k(\mathbf{V})}^G)_{r-2} = a^{\Lambda_{k-1}} Y_{2,k} (\varphi_{\mathbf{V}}^G)_{r-1} - a^{\Lambda_{r-1}} Y_{2,r}^2 (\varphi_{\mathbf{V}}^G)_{k-1}. \quad (7.12)$$

Substituting (7.11) and (7.12) for (7.10), it follows

$$\begin{aligned}
& (\varphi_{\mu_{r-1}\mu_{r-2}\dots\mu_k(\mathbf{V})}^G)_{r-1} \\
&= \frac{a^{2\Lambda_r}(Y_{1,r}^2 + 2\frac{a^{-\Lambda_{r-1}}Y_{1,r}}{Y_{2,r}}(\varphi_{\mathbf{V}}^G)_{r-1} + \frac{a^{-2\Lambda_{r-1}}}{Y_{2,r}^2}(\varphi_{\mathbf{V}}^G)_{r-1}^2)(a^{\Lambda_{k-1}}Y_{2,k}(\varphi_{\mathbf{V}}^G)_{r-1} - a^{\Lambda_{r-1}}Y_{2,r}^2(\varphi_{\mathbf{V}}^G)_{k-1})}{(\varphi_{\mathbf{V}}^G)_{r-1}} \\
& \quad + \frac{a^{2\Lambda_r+\Lambda_{r-1}}Y_{1,r}^2Y_{2,r}^2(\varphi_{\mathbf{V}}^G)_{k-1}}{(\varphi_{\mathbf{V}}^G)_{r-1}} \\
&= a^{2\Lambda_r+\Lambda_{k-1}}Y_{1,r}^2Y_{2,k} + 2a^{2\Lambda_r-\Lambda_{r-1}+\Lambda_{k-1}}\frac{Y_{2,k}Y_{1,r}}{Y_{2,r}}(\varphi_{\mathbf{V}}^G)_{r-1} \tag{7.13}
\end{aligned}$$

$$\begin{aligned}
& + a^{2\Lambda_r-2\Lambda_{r-1}+\Lambda_{k-1}}\frac{Y_{2,k}}{Y_{2,r}^2}(\varphi_{\mathbf{V}}^G)_{r-1}^2 - 2a^{2\Lambda_r}Y_{1,r}Y_{2,r}(\varphi_{\mathbf{V}}^G)_{k-1} - a^{2\Lambda_r-\Lambda_{r-1}}(\varphi_{\mathbf{V}}^G)_{r-1}(\varphi_{\mathbf{V}}^G)_{k-1} \\
&= a^{2\Lambda_r+\Lambda_{k-1}}\left(Y_{1,r}^2Y_{2,k} + 2\frac{Y_{2,k}Y_{1,r}Y_{1,r-1}}{Y_{2,r}}(1 + A_{1,r-1}^{-1} + \dots + A_{1,r-1}^{-1}\dots A_{1,k+1}^{-1}) \right. \\
& \quad \left. + \frac{Y_{2,k}Y_{1,r-1}^2}{Y_{2,r}^2}(1 + A_{1,r-1}^{-1} + \dots + A_{1,r-1}^{-1}\dots A_{1,1}^{-1})(1 + A_{1,r-1}^{-1} + \dots + A_{1,r-1}^{-1}\dots A_{1,k+1}^{-1})\right) \tag{7.14} \\
&= a^{2\Lambda_r+\Lambda_{k-1}}\left(Y_{1,r}^2Y_{2,k} + 2Y_{1,r}^2Y_{2,k}A_{1,r}^{-1}(1 + A_{1,r-1}^{-1} + \dots + A_{1,r-1}^{-1}\dots A_{1,k+1}^{-1}) \right. \\
& \quad \left. + Y_{1,r}^2Y_{2,k}A_{1,r}^{-2}(1 + A_{1,r-1}^{-1} + \dots + A_{1,r-1}^{-1}\dots A_{1,k+1}^{-1})^2 \right. \\
& \quad \left. + \frac{Y_{2,k}Y_{1,r-1}^2}{Y_{2,r}^2}A_{1,r-1}^{-1}\dots A_{1,k+1}^{-1}A_{1,k}^{-1}(1 + A_{1,r-1}^{-1} + \dots + A_{1,r-1}^{-1}\dots A_{1,k+1}^{-1}) \right. \\
& \quad \left. \times (1 + A_{1,k-1}^{-1} + A_{1,k-1}^{-1}A_{1,k-2}^{-1} + \dots + A_{1,k-1}^{-1}\dots A_{1,1}^{-1})\right),
\end{aligned}$$

where we use (6.5) in the third equality. Note that by Theorem 5.4, the monomial realization μ of the Demazure crystal $B(2\Lambda_r + \Lambda_k)_{s_{k+1}\dots s_{r-1}s_r}$ such that the highest weight vector is mapped to $Y := Y_{1,r}^2Y_{2,k}$ is as follows:

$$\begin{array}{ccccccc}
Y & \xrightarrow{r} & A_{1,r}^{-1}Y & \xrightarrow{r-1} & A_{1,r-1}^{-1}A_{1,r}^{-1}Y & \xrightarrow{r-2} & \dots \xrightarrow{k+1} A_{1,k+1}^{-1}\dots A_{1,r-1}^{-1}A_{1,r}^{-1}Y \\
& & \downarrow r & & & & \\
& & A_{1,r}^{-2}Y & \xrightarrow{r-1} & A_{1,r-1}^{-1}A_{1,r}^{-2}Y & \xrightarrow{r-2} & \dots \xrightarrow{k+1} A_{1,k+1}^{-1}\dots A_{1,r-1}^{-1}A_{1,r}^{-2}Y \\
& & & & \downarrow r-1 & & \\
& & & & A_{1,r-1}^{-2}A_{1,r}^{-2}Y & \xrightarrow{r-2} & \dots \xrightarrow{k+1} A_{1,k+1}^{-1}\dots A_{1,r-2}^{-1}A_{1,r-1}^{-2}A_{1,r}^{-2}Y \\
& & & & & & \downarrow r-2 \\
& & & & & & \dots \\
& & & & & & \downarrow k+1 \\
& & & & & & A_{1,k+1}^{-2}\dots A_{1,r-2}^{-2}A_{1,r-1}^{-2}A_{1,r}^{-2}Y
\end{array}$$

Thus, we see that

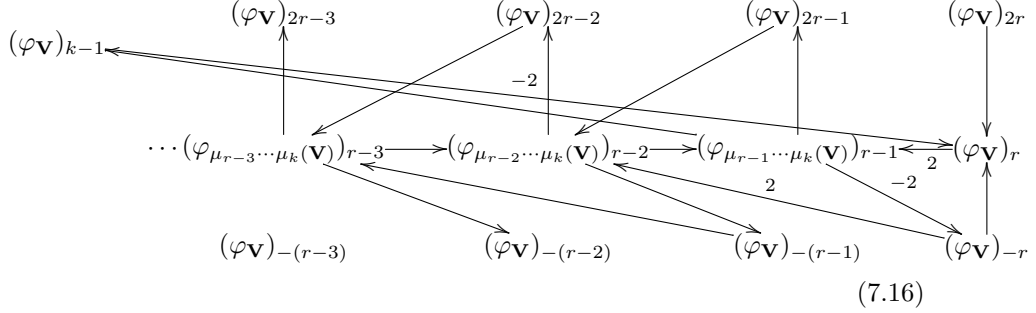
$$\begin{aligned}
& a^{2\Lambda_r+\Lambda_{k-1}}(Y_{1,r}^2Y_{2,k} + 2Y_{1,r}^2Y_{2,k}A_{1,r}^{-1}(1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1}A_{1,r-2}^{-1} + \dots + A_{1,r-1}^{-1}\dots A_{1,k+1}^{-1})) \\
& \quad + a^{2\Lambda_r+\Lambda_{k-1}}Y_{1,r}^2Y_{2,k}A_{1,r}^{-2}(1 + A_{1,r-1}^{-1} + \dots + A_{1,r-1}^{-1}\dots A_{1,k+1}^{-1})^2 \\
&= a^{2\Lambda_r+\Lambda_{k-1}} \sum_{b \in B(\Lambda_k+2\Lambda_r)_{s_{k+1}s_{k+2}\dots s_r}} C(b)\mu(b), \tag{7.15}
\end{aligned}$$

where each coefficient $C(b)$ is either 1 or 2. Similarly, we also see that

$$\begin{aligned}
& a^{2\Lambda_r + \Lambda_{k-1}} \frac{Y_{2,k} Y_{1,r-1}^2}{Y_{2,r}^2} A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1} A_{1,k}^{-1} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\
& \quad \times (1 + A_{1,k-1}^{-1} + A_{1,k-1}^{-1} A_{1,k-2}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,1}^{-1}), \\
& = a^{2\Lambda_r + \Lambda_{k-1}} Y_{1,r-1} Y_{1,k-1} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\
& \quad \times (1 + A_{1,k-1}^{-1} + A_{1,k-1}^{-1} A_{1,k-2}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,1}^{-1}), \\
& = a^{2\Lambda_r + \Lambda_{k-1}} \sum_{b \in B(\Lambda_{k-1} + \Lambda_{r-1})} \mu'(b),
\end{aligned}$$

where μ' is the monomial realization of the crystal base $B(\Lambda_{k-1} + \Lambda_{r-1})$ in our claim. Hence, we obtain (6.8).

Finally, we prove (6.10). Applying the mutation μ_{r-1} to the diagram (7.9), we obtain the mutation diagram of $\mu_{r-1} \mu_{r-2} \cdots \mu_{k+1} \mu_k(\Sigma_0)$:



This diagram says that

$$(\varphi_{\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_r = \frac{1}{(\varphi_{\mathbf{V}}^G)_r} \left((\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-1} + (\varphi_{\mathbf{V}}^G)_{k-1} (\varphi_{\mathbf{V}}^G)_{-r} (\varphi_{\mathbf{V}}^G)_{2r} \right). \quad (7.17)$$

Using (7.11) and (7.13), the following holds:

$$(\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-1} = a^{\Lambda_{k-1}} (\varphi_{\mathbf{V}}^G)_r^2 Y_{2,k} - a^{2\Lambda_r} Y_{1,r} Y_{2,r} (\varphi_{\mathbf{V}}^G)_{k-1} - a^{\Lambda_r} Y_{2,r} (\varphi_{\mathbf{V}}^G)_r (\varphi_{\mathbf{V}}^G)_{k-1}.$$

Applying this to (7.17), it is easy to see that

$$\begin{aligned}
& (\varphi_{\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_r \\
& = \frac{1}{(\varphi_{\mathbf{V}}^G)_r} (a^{\Lambda_{k-1}} Y_{2,k} (\varphi_{\mathbf{V}}^G)_r^2 - a^{\Lambda_r} Y_{2,r} (\varphi_{\mathbf{V}}^G)_r (\varphi_{\mathbf{V}}^G)_{k-1}) \\
& = a^{\Lambda_{k-1}} Y_{2,k} (\varphi_{\mathbf{V}}^G)_r - a^{\Lambda_r} Y_{2,r} (\varphi_{\mathbf{V}}^G)_{k-1} \\
& = a^{\Lambda_{k-1} + \Lambda_r} Y_{2,k} Y_{1,r} (1 + A_{1,r}^{-1} + A_{1,r-1}^{-1} A_{1,r-1}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}),
\end{aligned}$$

which means (6.10). \square

[Proof of Theorem 6.3 (iv).]

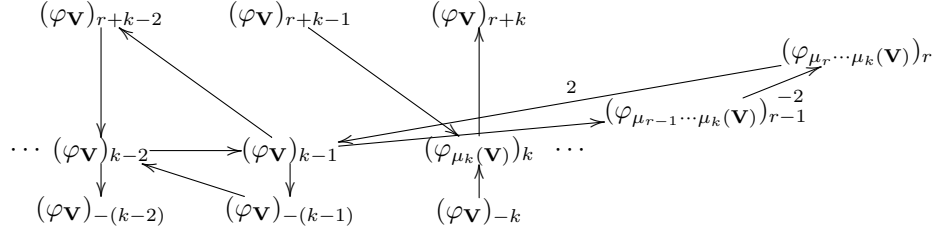
We put $A := 1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}$. It

now follows at once from (7.14) and (6.9) that

$$\begin{aligned}
& (\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k(\mathbf{V})})_{r-1} \\
&= a^{2\Lambda_r + \Lambda_{k-1}} Y_{2,k} (Y_{1,r}^2 + 2 \frac{Y_{1,r-1} Y_{1,r}}{Y_{2,r}} A + \frac{Y_{1,r-1}^2}{Y_{2,r}^2} A (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,1}^{-1})) \quad (7.18) \\
&= a^{2\Lambda_r + \Lambda_{k-1}} Y_{2,k} (Y_{1,r}^2 + 2 \frac{Y_{1,r-1} Y_{1,r}}{Y_{2,r}} A) \\
&+ a^{2\Lambda_r + \Lambda_{k-1}} \frac{Y_{2,k} Y_{1,r-1}^2}{Y_{2,r}^2} (A^2 + A \cdot A_{1,r-1}^{-1} \cdots A_{1,k}^{-1} (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} A_{1,k-2}^{-1} \cdots A_{1,1}^{-1})) \\
&= a^{2\Lambda_r + \Lambda_{k-1}} Y_{2,k} (Y_{1,r}^2 + 2 \frac{Y_{1,r-1} Y_{1,r}}{Y_{2,r}} A + \frac{Y_{1,r-1}^2}{Y_{2,r}^2} A^2 + a^{-\Lambda_{k-1}} \frac{Y_{1,r-1}}{Y_{2,k}} A (\varphi_{\mathbf{V}}^G)_{k-1}), \quad (7.19)
\end{aligned}$$

$$(\varphi_{\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k(\mathbf{V})})_r = a^{\Lambda_{k-1} + \Lambda_r} Y_{2,k} Y_{1,r} (1 + A \times A_{1,r}^{-1}). \quad (7.20)$$

To prove (6.11), we use the induction on $(k-j)$. First, we consider the case $k-j=1$. Applying the mutation μ_r to (7.16), one can verify that the vertices and arrows around $(\varphi_{\mathbf{V}})_{k-1}$ in the mutation diagram of $\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k(\Sigma_0)$ are as follows (Lemma 4.11):



Therefore, the cluster variable $(\varphi_{\mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k(\mathbf{V})})_{k-1}$ is

$$\frac{(\varphi_{\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k(\mathbf{V})})_r^2 (\varphi_{\mathbf{V}}^G)_{k-2} + (\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k(\mathbf{V})})_{r-1} (\varphi_{\mathbf{V}}^G)_{r+k-2} (\varphi_{\mathbf{V}}^G)_{-(k-1)}}{(\varphi_{\mathbf{V}}^G)_{k-1}}. \quad (7.21)$$

Substituting (7.19), (7.20) and $(\varphi_{\mathbf{V}}^G)_{k-2} = a^{\Lambda_{k-2} - \Lambda_{k-1}} \frac{Y_{2,k-1}}{Y_{2,k}} (\varphi_{\mathbf{V}}^G)_{k-1} - a^{\Lambda_{k-2}} \frac{Y_{1,k-1} Y_{2,k-1}}{Y_{2,k}}$

(7.6) for (7.21), we get

$$\begin{aligned}
& (\varphi_{\mu_{k-1}\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k(\mathbf{V})}^G)_{k-1} \\
&= \frac{a^{2\Lambda_{k-1}+2\Lambda_r}Y_{2,k}^2Y_{1,r}^2(1+A \times A_{1,r}^{-1})^2(a^{\Lambda_{k-2}-\Lambda_{k-1}}\frac{Y_{2,k-1}}{Y_{2,k}}(\varphi_{\mathbf{V}}^G)_{k-1} - a^{\Lambda_{k-2}}\frac{Y_{1,k-1}Y_{2,k-1}}{Y_{2,k}})}{(\varphi_{\mathbf{V}}^G)_{k-1}} \\
&+ \frac{a^{2\Lambda_r+2\Lambda_{k-1}+\Lambda_{k-2}}Y_{1,k-1}Y_{2,k-1}Y_{2,k}(Y_{1,r}^2 + 2\frac{Y_{1,r-1}Y_{1,r}}{Y_{2,r}}A + \frac{Y_{1,r-1}^2}{Y_{2,r}^2}A^2 + a^{-\Lambda_{k-1}}\frac{Y_{1,r-1}}{Y_{2,k}}A(\varphi_{\mathbf{V}}^G)_{k-1})}{(\varphi_{\mathbf{V}}^G)_{k-1}} \\
&= \frac{a^{2\Lambda_{k-1}+2\Lambda_r}Y_{2,k}^2Y_{1,r}^2(1+A \times A_{1,r}^{-1})^2(a^{\Lambda_{k-2}-\Lambda_{k-1}}\frac{Y_{2,k-1}}{Y_{2,k}}(\varphi_{\mathbf{V}}^G)_{k-1})}{(\varphi_{\mathbf{V}}^G)_{k-1}} \\
&+ \frac{a^{2\Lambda_r+2\Lambda_{k-1}+\Lambda_{k-2}}Y_{1,k-1}Y_{2,k-1}Y_{2,k}(a^{-\Lambda_{k-1}}\frac{Y_{1,r-1}}{Y_{2,k}}A(\varphi_{\mathbf{V}}^G)_{k-1})}{(\varphi_{\mathbf{V}}^G)_{k-1}} \\
&= a^{2\Lambda_r+\Lambda_{k-1}+\Lambda_{k-2}}(Y_{2,k-1}Y_{2,k}Y_{1,r}^2(1+A \times A_{1,r}^{-1})^2 + Y_{1,k-1}Y_{2,k-1}Y_{1,r-1}A) \\
&= a^{2\Lambda_r+\Lambda_{k-1}+\Lambda_{k-2}}(Y_{2,k-1}Y_{2,k}Y_{1,r}^2 + 2\frac{Y_{2,k-1}Y_{2,k}Y_{1,r-1}Y_{1,r}}{Y_{2,r}}A) \\
&+ a^{2\Lambda_r+\Lambda_{k-1}+\Lambda_{k-2}}(\frac{Y_{2,k-1}Y_{2,k}Y_{1,r-1}^2}{Y_{2,r}^2}A^2 + Y_{1,k-1}Y_{2,k-1}Y_{1,r-1}A) \\
&= a^{2\Lambda_r+\Lambda_{k-1}+\Lambda_{k-2}}(Y_{2,k-1}Y_{2,k}Y_{1,r}^2 + 2\frac{Y_{2,k-1}Y_{2,k}Y_{1,r-1}Y_{1,r}}{Y_{2,r}}A) \\
&+ a^{2\Lambda_r+\Lambda_{k-1}+\Lambda_{k-2}}\frac{Y_{2,k-1}Y_{2,k}Y_{1,r-1}^2}{Y_{2,r}^2}A(A + A_{1,r-1}^{-1}A_{1,r-2}^{-1}\cdots A_{1,k}^{-1}).
\end{aligned}$$

Thus, we obtain (6.11) for $j = k - 1$.

Next, we consider the case $k - j > 1$. The vertices and arrows around $(\varphi_{\mathbf{V}})_j$ in the mutation diagram of $\mu_{j+1}\cdots\mu_{k-2}\mu_{k-1}\mu_r\cdots\mu_{k+1}\mu_k(\Sigma_0)$ are as follows:

$$\begin{array}{ccccccc}
& (\varphi_{\mathbf{V}})_{r+j-1} & & (\varphi_{\mathbf{V}})_{r+j} & & (\varphi_{\mathbf{V}})_{r+j+1} & & (\varphi_{\mu_{r-1}\cdots\mu_k(\mathbf{V})})_{r-1} \\
& \downarrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow \\
\cdots & (\varphi_{\mathbf{V}})_{j-1} & \longrightarrow & (\varphi_{\mathbf{V}})_j & \longleftarrow & (\varphi_{\mu_{j+1}\cdots\mu_{k-1}\mu_r\cdots\mu_k(\mathbf{V})})_{j+1} & \cdots & \\
& \downarrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow \\
& (\varphi_{\mathbf{V}})_{-(j-1)} & & (\varphi_{\mathbf{V}})_{-j} & & (\varphi_{\mathbf{V}})_{-(j+1)} & &
\end{array}
\tag{7.22}$$

By this diagram and the induction hypothesis,

$$\begin{aligned}
& (\varphi_{\mu_j \cdots \mu_{k-2} \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_j \\
&= \frac{(\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_{r-1} (\varphi_{\mathbf{V}}^G)_{r+j-1} (\varphi_{\mathbf{V}}^G)_{-j} + (\varphi_{\mathbf{V}}^G)_{j-1} (\varphi_{\mu_{j+1} \cdots \mu_{k-2} \mu_{k-1} \mu_r \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_{j+1}}{(\varphi_{\mathbf{V}}^G)_j} \\
&= \frac{a^{\Lambda_{j-1} + \Lambda_j + 2\Lambda_r + \Lambda_{k-1}} Y_{1,j} Y_{2,j} Y_{2,k}}{(\varphi_{\mathbf{V}}^G)_j} \left(Y_{1,r}^2 + 2 \frac{Y_{1,r-1} Y_{1,r}}{Y_{2,r}} A \right. \\
&+ \left. \frac{Y_{1,r-1}^2}{Y_{2,r}^2} A (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,1}^{-1}) \right) + \frac{1}{(\varphi_{\mathbf{V}}^G)_j} (a^{\Lambda_{j-1} - \Lambda_j} \frac{Y_{2,j} (\varphi_{\mathbf{V}}^G)_j}{Y_{2,j+1}} - a^{\Lambda_{j-1}} \frac{Y_{1,j} Y_{2,j}}{Y_{2,j+1}}) \\
&\times a^{\Lambda_j + \Lambda_{k-1} + 2\Lambda_r} Y_{2,k} Y_{2,j+1} (Y_{1,r}^2 + 2 \frac{Y_{1,r-1} Y_{1,r}}{Y_{2,r}} A + \frac{Y_{1,r-1}^2}{Y_{2,r}^2} A (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,j+2}^{-1})) \\
&= a^{\Lambda_{j-1} + 2\Lambda_r + \Lambda_{k-1}} Y_{2,j} Y_{2,k} (Y_{1,r}^2 + 2 \frac{Y_{1,r-1} Y_{1,r}}{Y_{2,r}} A + \frac{Y_{1,r-1}^2}{Y_{2,r}^2} A (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,j+2}^{-1})) \\
&+ \frac{a^{\Lambda_{j-1} + 2\Lambda_r + \Lambda_{k-1}} Y_{2,j} Y_{2,k}}{(\varphi_{\mathbf{V}}^G)_j} a^{\Lambda_j} \frac{Y_{1,j} Y_{1,r-1}^2}{Y_{2,r}^2} A \cdot A_{1,r-1}^{-1} \cdots A_{1,j+1}^{-1} (1 + A_{1,j}^{-1} + \cdots + A_{1,j}^{-1} \cdots A_{1,1}^{-1}) \\
&= a^{\Lambda_{j-1} + 2\Lambda_r + \Lambda_{k-1}} Y_{2,j} Y_{2,k} (Y_{1,r}^2 + 2 \frac{Y_{1,r-1} Y_{1,r}}{Y_{2,r}} A + \frac{Y_{1,r-1}^2}{Y_{2,r}^2} A (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,j+1}^{-1})).
\end{aligned}$$

Hence, we obtain (6.11). By using (6.11), it is easy to see that

$$\begin{aligned}
& a^{-(\Lambda_{j-1} + \Lambda_{k-1} + 2\Lambda_r)} (\varphi_{\mu_j \cdots \mu_{k-2} \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_j(a; \mathbf{Y}) \\
&= Y_{2,k} Y_{2,j} Y_{1,r}^2 + 2 Y_{2,k} Y_{2,j} Y_{1,r}^2 A_{1,r}^{-1} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\
&+ Y_{2,k} Y_{2,j} Y_{1,r}^2 A_{1,r}^{-2} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1})^2 \\
&Y_{2,k} Y_{2,j} Y_{1,r}^2 A_{1,r}^{-1} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\
&\quad \times A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,k}^{-1} (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,j+1}^{-1}).
\end{aligned}$$

Just as in the proof of (7.15), we obtain

$$\begin{aligned}
& Y_{2,k} Y_{2,j} Y_{1,r}^2 + 2 Y_{2,k} Y_{2,j} Y_{1,r}^2 A_{1,r}^{-1} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\
&+ Y_{2,k} Y_{2,j} Y_{1,r}^2 A_{1,r}^{-2} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1})^2 \\
&= \sum_{b \in B(\Lambda_j + \Lambda_k + 2\Lambda_r)_{s_{k+1} s_{k+2} \cdots s_r}} C(b) \mu(b),
\end{aligned}$$

where μ is the monomial realization in our claim, and each coefficient $C(b)$ is either 1 or 2. Similarly, we also obtain

$$\begin{aligned}
& Y_{2,k} Y_{2,j} Y_{1,r}^2 A_{1,r}^{-2} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\
&\quad \times A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,k}^{-1} (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,j+1}^{-1}) \\
&= Y_{1,k-1} Y_{2,j} Y_{1,r-1} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\
&\quad \times (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,j+1}^{-1}) \\
&= \sum_{b \in B(\Lambda_j + \Lambda_{k-1} + \Lambda_{r-1})_{s_{j+1} \cdots s_{k-1} s_{k+1} s_{k+2} \cdots s_{r-1}}} \mu'(b).
\end{aligned}$$

Therefore, we have (6.12). \square

7.2 The proof of Theorem 6.4

Next, let us prove Theorem 6.4. It will be proved in a similar way to Theorem 6.3. Let $G = \text{Sp}_{2r}$. First, we set the Laurent monomials as follows:

$$C(l, k) := \begin{cases} \frac{Y_{l,k}}{Y_{l,k-1}} & \text{if } 1 \leq k \leq r, \\ \frac{Y_{l,|k|-1}}{Y_{l+1,|k|}} & \text{if } \bar{r} \leq k \leq \bar{1}, \end{cases} \quad (7.23)$$

where for $1 \leq l \leq r$, set $|l| = |\bar{l}| = l$. The mutation diagram (Definition 4.10) of the initial seed Σ_0 is

$$\begin{array}{ccccccccccccccc} (\varphi \mathbf{V})_{r+1} & & (\varphi \mathbf{V})_{r+k-1} & & (\varphi \mathbf{V})_{r+k} & & (\varphi \mathbf{V})_{r+k+1} & & (\varphi \mathbf{V})_{2r-1} & & -2 & & (\varphi \mathbf{V})_{2r} \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & 2 & \nearrow & \downarrow \\ (\varphi \mathbf{V})_1 & \longrightarrow & \cdots & (\varphi \mathbf{V})_{k-1} & \longrightarrow & (\varphi \mathbf{V})_k & \longrightarrow & (\varphi \mathbf{V})_{k+1} & \longrightarrow & \cdots & (\varphi \mathbf{V})_{r-1} & \longrightarrow & (\varphi \mathbf{V})_r \\ \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow & \nwarrow & -2 & \nwarrow & \downarrow \\ (\varphi \mathbf{V})_{-1} & & (\varphi \mathbf{V})_{-(k-1)} & & (\varphi \mathbf{V})_{-k} & & (\varphi \mathbf{V})_{-(k+1)} & & (\varphi \mathbf{V})_{-(r-1)} & & & & (\varphi \mathbf{V})_{-r} \end{array} \quad (7.24)$$

Proposition 7.3. (i) For $k \in [1, r]$, the initial cluster variables $(\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y})$ in $\mathbb{C}[G^{e,c^2}]$ are described as

$$(\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y}) = a^{\Lambda_k} \sum_{b \in B(\Lambda_k)_{c^2_{>2r-k}}} C(b) \mu(b),$$

where $\mu : B(\Lambda_k) \rightarrow \mathcal{Y}$ is the monomial realization of $B(\Lambda_k)$ in Theorem 6.4 (i), $C(b)$ are some positive integers.

(ii) For $k \in [1, r]$, the frozen cluster variables $(\varphi_{\mathbf{V}}^G)_{-k}(a; \mathbf{Y})$, $(\varphi_{\mathbf{V}}^G)_{r+k}(a; \mathbf{Y})$ in $\mathbb{C}[G^{e,c^2}]$ are described as

$$(\varphi_{\mathbf{V}}^G)_{-k}(a; \mathbf{Y}) = a^{\Lambda_k} Y_{1,k} Y_{2,k}, \quad (\varphi_{\mathbf{V}}^G)_{r+k}(a; \mathbf{Y}) = a^{\Lambda_k}.$$

[Proof.]

For $1 \leq k \leq r-1$, we can prove

$$\begin{aligned} (\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y}) &= a^{\Lambda_k} Y_{1,k} (1 + A_{1,k}^{-1} + A_{1,k}^{-1} A_{1,k-1}^{-1} + \cdots + A_{1,k}^{-1} \cdots A_{1,1}^{-1}) \\ &= a^{\Lambda_k} \sum_{b \in B(\Lambda_k)_{c^2_{>2r-k}}} \mu(b), \end{aligned} \quad (7.25)$$

by the same argument as in Proposition 7.1.

Next, let us consider the case $k = r$. Just as in (7.2), it follows

$$\overline{c_{>r}^2} v_1 \wedge \cdots \wedge v_r = \overline{s_1} \cdots \overline{s_r} (v_1 \wedge \cdots \wedge v_r) = v_2 \wedge \cdots \wedge v_r \wedge v_{\bar{1}}.$$

Using the bilinear form in 4.3, the cluster variable $(\varphi_{\mathbf{V}}^G)_r(a; \mathbf{Y})$ is described as

$$\begin{aligned} a^{\Lambda_r} \langle v_2 \wedge \cdots \wedge v_r \wedge v_{\bar{1}}, \alpha_1^\vee(Y_{2,1}) y_1(Y_{2,1}) \cdots \alpha_r^\vee(Y_{2,r}) y_r(Y_{2,r}) \\ \alpha_1^\vee(Y_{1,1}) y_1(Y_{1,1}) \cdots \alpha_r^\vee(Y_{1,r}) y_r(Y_{1,r}) v_1 \wedge \cdots \wedge v_r \rangle. \end{aligned}$$

Using (2.3), (2.4) and (2.5) repeatedly, one obtain

$$\alpha_1^\vee(Y_{1,1}) y_1(Y_{1,1}) \cdots \alpha_r^\vee(Y_{1,r}) y_r(Y_{1,r}) v_i = \begin{cases} \frac{Y_{1,i}}{Y_{1,i-1}} v_i + v_{i+1} & \text{if } 1 \leq i \leq r-1, \\ \frac{Y_{1,r}}{Y_{1,r-1}} v_r + \sum_{j=1}^r Y_{1,j-1} v_{\bar{j}} & \text{if } i = r, \end{cases}$$

which means that $\alpha_1^\vee(Y_{1,1})y_1(Y_{1,1}) \cdots \alpha_r^\vee(Y_{1,r})y_r(Y_{1,r})v_1 \wedge \cdots \wedge v_r$ is a linear combination of $v_1 \wedge \cdots \wedge v_r$ and $v_1 \wedge \cdots \wedge v_s \wedge v_{s+2} \wedge \cdots \wedge v_r \wedge v_{\overline{t+1}}$ ($0 \leq s, t \leq r-1$) with the coefficients $C(1,1)C(1,2) \cdots C(1,r)$ and $C(1,1)C(1,2) \cdots C(1,s)Y_{1,t}$, respectively. Similarly, we see that the coefficient of $v_2 \wedge \cdots \wedge v_r \wedge v_{\overline{1}}$ in $\alpha_1^\vee(Y_{2,1})y_1(Y_{2,1}) \cdots \alpha_r^\vee(Y_{2,r})y_r(Y_{2,r})v_1 \wedge \cdots \wedge v_r$ is 1, and the coefficient in $\alpha_1^\vee(Y_{2,1})y_1(Y_{2,1}) \cdots \alpha_r^\vee(Y_{2,r})y_r(Y_{2,r})v_1 \wedge \cdots \wedge v_s \wedge v_{s+2} \wedge \cdots \wedge v_r \wedge v_{\overline{t+1}}$ is $C(2,s+2) \cdots C(2,r)Y_{2,t+1}^{-1}$. By the above argument, we obtain

$$\begin{aligned} & (\varphi_{\mathbf{V}}^G)_r(a; \mathbf{Y}) \\ &= a^{\Lambda_r}(Y_{1,r} + \sum_{0 \leq s, t \leq r-1} C(1,1)C(1,2) \cdots C(1,s)C(2,s+2) \cdots C(2,r)C(1,\overline{t+1})) \\ &= a^{\Lambda_r}(Y_{1,r} + Y_{1,r}A_{1,r}^{-1}(1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1}A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,1}^{-1})^2). \end{aligned} \quad (7.26)$$

From Theorem 5.4, the conclusion $(\varphi_{\mathbf{V}}^G)_r(a; \mathbf{Y}) = a^{\Lambda_r} \sum_{b \in B(\Lambda_k)_{c_{>r}^2}} C(b)\mu(b)$ follows, where each coefficient $C(b)$ is either 1 or 2. Thus, we obtain the claim (i). By the same argument, we can also obtain (ii). \square

By the same way as in (7.6), we see that for $k \in [1, r-2]$,

$$a^{-\Lambda_k}(\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y}) = Y_{1,k} + C(2,k+1)(a^{-\Lambda_{k-1}}(\varphi_{\mathbf{V}}^G)_{k-1}(a; \mathbf{Y})). \quad (7.27)$$

Lemma 7.4. *For $k \in [1, r]$, the cluster variables $(\varphi_{\mu_k \mu_{k+1} \cdots \mu_r(\mathbf{V})}^G)_k(a; \mathbf{Y})$ in $\mathbb{C}[G^{e,c^2}]$ are described as*

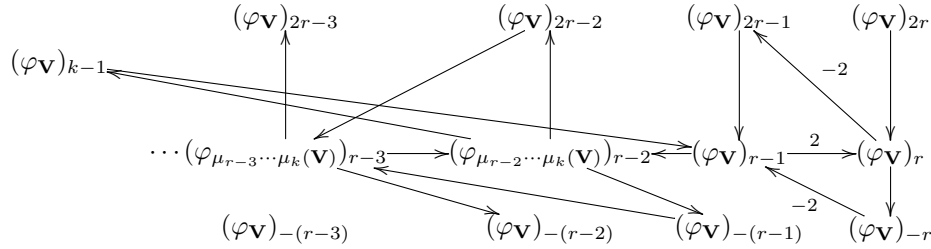
$$(\varphi_{\mu_k \mu_{k+1} \cdots \mu_r(\mathbf{V})}^G)_k(a; \mathbf{Y}) = a^{\Lambda_{k-1} + \Lambda_{r-1}} Y_{2,k}.$$

[Proof.]

Using the mutation diagram (7.24), we can prove this lemma in the same way as Lemma 7.2. \square

[Proof of Theorem 6.4 (ii) and (iii).]

Our claim (ii) is obtained by the same calculation as in Theorem 6.3 (ii). So let us consider the claim (iii). By Lemma 4.11, the mutation diagram of $\mu_{r-2} \cdots \mu_{k+1} \mu_k(\Sigma_0)$ is as follows:



From this diagram, we have

$$(\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_{r-1} = \frac{(\varphi_{\mathbf{V}}^G)_r(\varphi_{\mu_{r-2} \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_{r-2} + (\varphi_{\mathbf{V}}^G)_{k-1}(\varphi_{\mathbf{V}}^G)_{2r-1}(\varphi_{\mathbf{V}}^G)_{-r}}{(\varphi_{\mathbf{V}}^G)_{r-1}}. \quad (7.28)$$

Using the notation $A := (1 + A_{1,k-1}^{-1} + A_{1,k-1}^{-1}A_{1,k-2}^{-1} + \cdots + A_{1,k-1}^{-1}A_{1,k-2}^{-1} \cdots A_{1,1}^{-1})$, we can write

$$(\varphi_{\mathbf{V}}^G)_{k-1} = a^{\Lambda_{k-1}} Y_{1,k-1} A. \quad (7.29)$$

Note that by (7.25) and (6.13),

$$\begin{aligned}
& (\varphi_{\mu_{r-2} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-2}(a; \mathbf{Y}) \\
&= a^{\Lambda_{k-1} + \Lambda_{r-1}} Y_{2,k} (a^{-\Lambda_{r-1}} (\varphi_{\mathbf{V}}^G)_{r-1} - Y_{1,r-1} A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,k}^{-1} \cdot A), \\
&= a^{\Lambda_{k-1} + \Lambda_{r-1}} Y_{2,k} (a^{-\Lambda_{r-1}} (\varphi_{\mathbf{V}}^G)_{r-1} - \frac{Y_{1,k-1} Y_{2,r}}{Y_{2,k}} \cdot A) \tag{7.30}
\end{aligned}$$

and by (7.25) and (7.26),

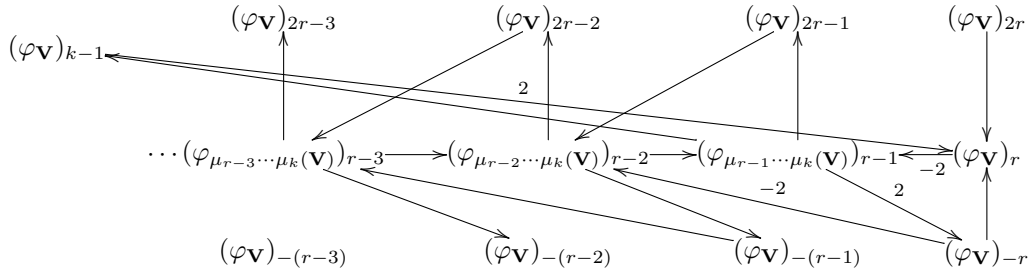
$$(\varphi_{\mathbf{V}}^G)_r = a^{\Lambda_r} (Y_{1,r} + a^{-2\Lambda_{r-1}} Y_{2,r}^{-1} (\varphi_{\mathbf{V}}^G)_{r-1}^2). \tag{7.31}$$

Substituting (7.29), (7.30) and (7.31) for (7.28), we get

$$\begin{aligned}
& (\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-1}(a; \mathbf{Y}) \\
&= \frac{a^{\Lambda_{k-1} + \Lambda_r} Y_{2,k} (\varphi_{\mathbf{V}}^G)_{r-1} (Y_{1,r} + a^{-2\Lambda_{r-1}} Y_{2,r}^{-1} (\varphi_{\mathbf{V}}^G)_{r-1}^2) - Y_{1,k-1} A a^{\Lambda_{k-1} - \Lambda_{r-1} + \Lambda_r} (\varphi_{\mathbf{V}}^G)_{r-1}^2}{(\varphi_{\mathbf{V}}^G)_{r-1}} \\
&= a^{\Lambda_{k-1} + \Lambda_r} Y_{2,k} (Y_{1,r} + a^{-2\Lambda_{r-1}} Y_{2,r}^{-1} (\varphi_{\mathbf{V}}^G)_{r-1}^2) - a^{\Lambda_{k-1} + \Lambda_r - \Lambda_{r-1}} Y_{1,k-1} A (\varphi_{\mathbf{V}}^G)_{r-1} \tag{7.32} \\
&= a^{\Lambda_{k-1} + \Lambda_r} Y_{2,k} Y_{1,r} \\
&+ a^{\Lambda_{k-1} + \Lambda_r - \Lambda_{r-1}} (\varphi_{\mathbf{V}}^G)_{r-1} \left(\frac{Y_{1,r-1} Y_{2,k}}{Y_{2,r}} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,1}^{-1}) - Y_{1,k-1} A \right) \\
&= a^{\Lambda_{k-1} + \Lambda_r} Y_{2,k} Y_{1,r} + a^{\Lambda_{k-1} + \Lambda_r} \frac{Y_{2,k} Y_{1,r}^2}{Y_{2,r}} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \tag{7.33} \\
&\quad \times (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,1}^{-1}) \\
&= a^{\Lambda_{k-1} + \Lambda_r} \left(Y_{2,k} Y_{1,r} + Y_{2,k} Y_{1,r} A_{1,r}^{-1} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1})^2 \right. \\
&\quad \left. + Y_{1,k-1} Y_{1,r-1} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,1}^{-1}) \right),
\end{aligned}$$

which implies (6.15).

Next, let us consider the cluster variable $(\varphi_{\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_r(a; \mathbf{Y})$. The mutation diagram of $\mu_{r-1} \cdots \mu_{k+1} \mu_k(\Sigma_0)$ is as follows:



Thus,

$$(\varphi_{\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_r = \frac{(\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-1}^2 + (\varphi_{\mathbf{V}}^G)_{k-1}^2 (\varphi_{\mathbf{V}}^G)_{2r} (\varphi_{\mathbf{V}}^G)_{-r}}{(\varphi_{\mathbf{V}}^G)_r}. \tag{7.34}$$

From (7.31) and (7.32), we see that

$$\begin{aligned}
& (\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-1}^2 \\
&= a^{2\Lambda_{k-1}} (Y_{2,k}(\varphi_{\mathbf{V}}^G)_r - a^{\Lambda_r - \Lambda_{r-1}} Y_{1,k-1} A(\varphi_{\mathbf{V}}^G)_{r-1})^2 \\
&= a^{2\Lambda_{k-1}} (Y_{2,k}^2(\varphi_{\mathbf{V}}^G)_r^2 - 2a^{\Lambda_r - \Lambda_{r-1}} Y_{2,k} Y_{1,k-1} A(\varphi_{\mathbf{V}}^G)_{r-1}(\varphi_{\mathbf{V}}^G)_r + a^{2\Lambda_r - 2\Lambda_{r-1}} Y_{1,k-1}^2 A^2(\varphi_{\mathbf{V}}^G)_{r-1}^2) \\
&= a^{2\Lambda_{k-1}} (\varphi_{\mathbf{V}}^G)_r (Y_{2,k}^2(\varphi_{\mathbf{V}}^G)_r - 2a^{\Lambda_r - \Lambda_{r-1}} Y_{2,k} Y_{1,k-1} A(\varphi_{\mathbf{V}}^G)_{r-1} + a^{\Lambda_r} Y_{1,k-1}^2 Y_{2,r} A^2) \\
&\quad - a^{2\Lambda_{k-1} + 2\Lambda_r} Y_{1,k-1}^2 Y_{1,r} Y_{2,r} A^2.
\end{aligned}$$

Since we know that $(\varphi_{\mathbf{V}}^G)_{k-1} = a^{\Lambda_{k-1}} Y_{1,k-1} A$ (7.25), from (7.34), one obtain

$$\begin{aligned}
& (\varphi_{\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_r \\
&= a^{2\Lambda_{k-1}} (Y_{2,k}^2(\varphi_{\mathbf{V}}^G)_r - 2a^{\Lambda_r - \Lambda_{r-1}} Y_{2,k} Y_{1,k-1} A(\varphi_{\mathbf{V}}^G)_{r-1} + a^{\Lambda_r} Y_{1,k-1}^2 Y_{2,r} A^2). \quad (7.35)
\end{aligned}$$

By (7.25) and (7.26), we get

$$\begin{aligned}
& Y_{2,k}^2(\varphi_{\mathbf{V}}^G)_r - a^{\Lambda_r - \Lambda_{r-1}} Y_{2,k} Y_{1,k-1} A(\varphi_{\mathbf{V}}^G)_{r-1} \\
&= a^{\Lambda_r} Y_{1,r} Y_{2,k}^2 + a^{\Lambda_r} Y_{1,r} Y_{2,k} A_{1,r}^{-1} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,1}^{-1}) \\
&\quad \times (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}),
\end{aligned}$$

and

$$\begin{aligned}
& -a^{\Lambda_r - \Lambda_{r-1}} Y_{2,k} Y_{1,k-1} A(\varphi_{\mathbf{V}}^G)_{r-1} + a^{\Lambda_r} Y_{1,k-1}^2 Y_{2,r} A^2 \\
&= -a^{\Lambda_r} Y_{2,k} Y_{1,k-1} Y_{1,r-1} A(1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}).
\end{aligned}$$

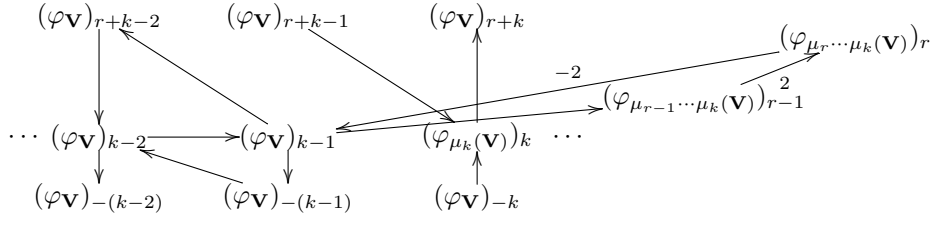
Thus, it follows

$$\begin{aligned}
& (\varphi_{\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_r \\
&= a^{\Lambda_r + 2\Lambda_{k-1}} Y_{1,r} Y_{2,k}^2 (1 + A_{1,r}^{-1} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1})^2).
\end{aligned}$$

From this explicit formula, the conclusion (6.17) follows. \square

[Proof of Theorem 6.4 (iv).]

To prove our claim, we use the induction on $(k-j)$. First, let $k-j=1$. It follows by Lemma 4.11 that the mutation diagram of $\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k(\Sigma_0)$ is



which means that

$$\begin{aligned}
& (\varphi_{\mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{k-1} \\
&= \frac{(\varphi_{\mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_r (\varphi_{\mathbf{V}}^G)_{k-2} + (\varphi_{\mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-1} (\varphi_{\mathbf{V}}^G)_{r+k-2} (\varphi_{\mathbf{V}}^G)_{-(k-1)}}{(\varphi_{\mathbf{V}}^G)_{k-1}}.
\end{aligned}$$

By (7.31) and (7.32), we get

$$(\varphi_{\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{r-1} = a^{\Lambda_{k-1}}Y_{2,k}(\varphi_{\mathbf{V}}^G)_r - a^{\Lambda_r-\Lambda_{r-1}}(\varphi_{\mathbf{V}}^G)_{k-1}(\varphi_{\mathbf{V}}^G)_{r-1},$$

and (7.35) implies that

$$\begin{aligned} (\varphi_{\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_r &= a^{2\Lambda_{k-1}}Y_{2,k}^2(\varphi_{\mathbf{V}}^G)_r \\ &- 2a^{\Lambda_r-\Lambda_{r-1}+\Lambda_{k-1}}Y_{2,k}(\varphi_{\mathbf{V}}^G)_{k-1}(\varphi_{\mathbf{V}}^G)_{r-1} + a^{\Lambda_r}Y_{2,r}(\varphi_{\mathbf{V}}^G)_{k-1}^2 = a^{\Lambda_{k-1}}Y_{2,k}(\varphi_{\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{r-1} \\ &- a^{\Lambda_r-\Lambda_{r-1}+\Lambda_{k-1}}Y_{2,k}(\varphi_{\mathbf{V}}^G)_{k-1}(\varphi_{\mathbf{V}}^G)_{r-1} + a^{\Lambda_r}Y_{2,r}(\varphi_{\mathbf{V}}^G)_{k-1}^2. \end{aligned} \quad (7.36)$$

As have seen in (7.27), it follows that

$$(\varphi_{\mathbf{V}}^G)_{k-2} = a^{\Lambda_{k-2}-\Lambda_{k-1}}\frac{Y_{2,k-1}}{Y_{2,k}}(\varphi_{\mathbf{V}}^G)_{k-1} - a^{\Lambda_{k-2}}\frac{Y_{1,k-1}Y_{2,k-1}}{Y_{2,k}}. \quad (7.37)$$

By using the formulas (7.36) and (7.37), we get

$$\begin{aligned} &(\varphi_{\mu_{k-1}\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{k-1} \\ &= a^{\Lambda_{k-2}-\Lambda_{k-1}}\frac{Y_{2,k-1}}{Y_{2,k}}(\varphi_{\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_r + a^{\Lambda_r-\Lambda_{r-1}+\Lambda_{k-1}+\Lambda_{k-2}}Y_{1,k-1}Y_{2,k-1}(\varphi_{\mathbf{V}}^G)_{r-1} \\ &- a^{\Lambda_r+\Lambda_{k-2}}\frac{Y_{1,k-1}Y_{2,k-1}Y_{2,r}}{Y_{2,k}}(\varphi_{\mathbf{V}}^G)_{k-1} \\ &= a^{\Lambda_r+\Lambda_{k-1}+\Lambda_{k-2}}Y_{2,k-1}Y_{2,k}Y_{1,r}(1 + A_{1,r}^{-1}(1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1}\cdots A_{1,k+1}^{-1})^2) \\ &+ a^{\Lambda_r+\Lambda_{k-1}+\Lambda_{k-2}}Y_{1,k-1}Y_{2,k-1}Y_{1,r-1}(1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1}\cdots A_{1,k+1}^{-1}) \\ &= a^{\Lambda_r+\Lambda_{k-1}+\Lambda_{k-2}}Y_{2,k-1}Y_{2,k}Y_{1,r} + a^{\Lambda_r+\Lambda_{k-1}+\Lambda_{k-2}}Y_{2,k-1}Y_{2,k}Y_{1,r}A_{1,r}^{-1} \\ &\quad \times (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1}\cdots A_{1,k+1}^{-1})(1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1}\cdots A_{1,k}^{-1}), \end{aligned}$$

where we use (7.25) and (6.16) in the second equality. Thus, we obtain our claim (6.18) for $j = k - 1$.

Next, we assume $k-j > 1$. The mutation diagram of $\mu_{j+1}\cdots\mu_{k-2}\mu_{k-1}\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k(\Sigma_0)$ is the same form as in (7.22), which means

$$\begin{aligned} &(\varphi_{\mu_j\mu_{j+1}\cdots\mu_{k-2}\mu_{k-1}\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_j(a; \mathbf{Y}) \\ &= \frac{(\varphi_{\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{r-1}(\varphi_{\mathbf{V}}^G)_{r+j-1}(\varphi_{\mathbf{V}}^G)_{-j} + (\varphi_{\mu_{j+1}\cdots\mu_{k-2}\mu_{k-1}\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{j+1}(\varphi_{\mathbf{V}}^G)_{j-1}}{(\varphi_{\mathbf{V}}^G)_j}. \end{aligned}$$

By the induction hypothesis, we get

$$\begin{aligned} &(\varphi_{\mu_{j+1}\cdots\mu_{k-2}\mu_{k-1}\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{j+1}(a; \mathbf{Y}) \\ &= a^{\Lambda_j+\Lambda_{k-1}+\Lambda_r}Y_{2,k}Y_{2,j+1}Y_{1,r} \\ &+ a^{\Lambda_j+\Lambda_{k-1}+\Lambda_r}\frac{Y_{2,k}Y_{2,j+1}Y_{1,r-1}^2}{Y_{2,r}}(1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1}A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1}\cdots A_{1,k+1}^{-1}) \\ &\quad \times (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1}A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1}\cdots A_{1,j+2}^{-1}). \end{aligned} \quad (7.38)$$

Since the equation $(\varphi_{\mathbf{V}}^G)_{j-1} = a^{\Lambda_{j-1}-\Lambda_j}\frac{Y_{2,j}}{Y_{2,j+1}}(\varphi_{\mathbf{V}}^G)_j - a^{\Lambda_{j-1}}\frac{Y_{1,j}Y_{2,j}}{Y_{2,j+1}}$ holds from

(7.27), it follows by (7.33) and (7.38) that

$$\begin{aligned}
& (\varphi_{\mu_j \mu_{j+1} \cdots \mu_{k-2} \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{v}))_j(a; \mathbf{Y}) \\
&= \frac{a^{\Lambda_{j-1} - \Lambda_j} \frac{Y_{2,j}}{Y_{2,j+1}} (\varphi_{\mu_{j+1} \cdots \mu_{k-2} \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{v}))_{j+1} (\varphi_{\mathbf{V}}^G)_j}{(\varphi_{\mathbf{V}}^G)_j} \\
&\quad + \frac{a^{\Lambda_{j-1} + \Lambda_j + \Lambda_{k-1} + \Lambda_r} Y_{1,j} Y_{2,j} Y_{2,k} Y_{1,r-1}^2}{(\varphi_{\mathbf{V}}^G)_j Y_{2,r}} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\
&\quad \times A_{1,r-1}^{-1} \cdots A_{1,j+2}^{-1} A_{1,j+1}^{-1} (1 + A_{1,j}^{-1} + \cdots + A_{1,j}^{-1} \cdots A_{1,1}^{-1}) \\
&= a^{\Lambda_{j-1} - \Lambda_j} \frac{Y_{2,j}}{Y_{2,j+1}} (\varphi_{\mu_{j+1} \cdots \mu_{k-2} \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{v}))_{j+1} \\
&\quad + a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_r} \frac{Y_{2,j} Y_{2,k} Y_{1,r-1}^2}{Y_{2,r}} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) A_{1,r-1}^{-1} \cdots A_{1,j+2}^{-1} A_{1,j+1}^{-1} \\
&= a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_r} Y_{2,j} Y_{2,k} Y_{1,r} + a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_r} \frac{Y_{2,j} Y_{2,k} Y_{1,r-1}^2}{Y_{2,r}} \\
&\quad \times (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,j+1}^{-1}),
\end{aligned}$$

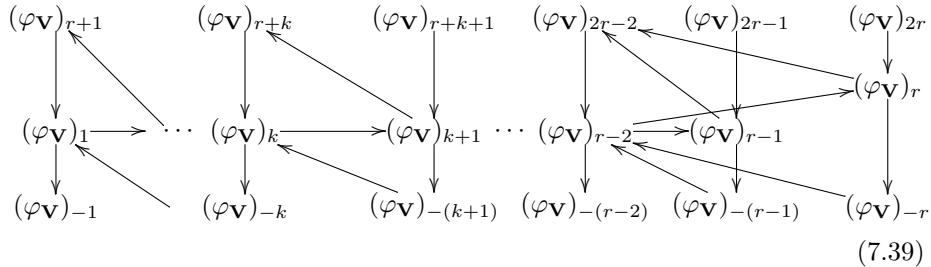
where we use (7.25) in the second equality, and use (7.38) in the third equality. Thus, we get (6.18), and one can rewrite it as follows:

$$\begin{aligned}
& (\varphi_{\mu_j \mu_{j+1} \cdots \mu_{k-2} \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k}^G(\mathbf{v}))_j(a; \mathbf{Y}) \\
&= a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_r} Y_{2,j} Y_{2,k} Y_{1,r} (1 + A_{1,r}^{-1} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1})^2) \\
&\quad + a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_r} Y_{2,j} Y_{1,k-1} Y_{1,r-1} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\
&\quad \times (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,j+1}^{-1}) \\
&= a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_r} \sum_{b \in B(\Lambda_j + \Lambda_k + 2\Lambda_r)_{s_{k+1} s_{k+2} \cdots s_r}} C(b) \mu(b) \\
&\quad + a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_r} \sum_{b \in B(\Lambda_j + \Lambda_{k-1} + \Lambda_{r-1})_{s_{j+1} \cdots s_{k-1} s_{k+1} s_{k+2} \cdots s_{r-1}}} \mu'(b),
\end{aligned}$$

where each coefficient $C(b)$ is either 1 or 2. Thus, we obtain (6.19) in the same way as (7.15). \square

7.3 The proof of Theorem 6.5

In this final subsection, we shall prove Theorem 6.5. Let $G = \mathrm{SO}_{2r}$. The mutation diagram of the initial seed Σ_0 is



First, we can prove the following proposition in a similar way to Proposition 7.1 and 7.3.

Proposition 7.5. (i) For $k \in [1, r]$, the initial cluster variables $(\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y})$ in $\mathbb{C}[G^{e, c^2}]$ are described as

$$(\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y}) = a^{\Lambda_k} \sum_{b \in B(\Lambda_k)_{c^2} > 2r-k} \mu(b),$$

where $\mu : B(\Lambda_k) \rightarrow \mathcal{Y}$ is the monomial realization of $B(\Lambda_k)$ in Theorem 6.5 (i).

(ii) For $k \in [1, r]$, the frozen cluster variables $(\varphi_{\mathbf{V}}^G)_{-k}(a; \mathbf{Y})$, $(\varphi_{\mathbf{V}}^G)_{r+k}(a; \mathbf{Y})$ in $\mathbb{C}[G^{e, c^2}]$ are described as

$$(\varphi_{\mathbf{V}}^G)_{-k}(a; \mathbf{Y}) = a^{\Lambda_k} Y_{1,k} Y_{2,k}, \quad (\varphi_{\mathbf{V}}^G)_{r+k}(a; \mathbf{Y}) = a^{\Lambda_k}.$$

For $k \in [1, r-1]$, just as in (6.5), (7.5) and (7.25), we see that

$$(\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y}) = a^{\Lambda_k} Y_{1,k} (1 + A_{1,k}^{-1} + A_{1,k}^{-1} A_{1,k-1}^{-1} + \cdots + A_{1,k}^{-1} \cdots A_{1,1}^{-1}), \quad (7.40)$$

and

$$(\varphi_{\mathbf{V}}^G)_r(a; \mathbf{Y}) = a^{\Lambda_r} Y_{1,r} (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,1}^{-1}). \quad (7.41)$$

Considering (7.40), we see that for $k \in [1, r-2]$,

$$a^{-\Lambda_k} (\varphi_{\mathbf{V}}^G)_k(a; \mathbf{Y}) = Y_{1,k} + \frac{Y_{2,k+1}}{Y_{2,k}} (a^{-\Lambda_{k-1}} (\varphi_{\mathbf{V}}^G)_{k-1}(a; \mathbf{Y})). \quad (7.42)$$

Lemma 7.6. (i)

$$(\varphi_{\mu_{r-1}(\mathbf{V})}^G)_{r-1}(a; \mathbf{Y}) = a^{\Lambda_{r-2}} Y_{2,r-1}, \quad (\varphi_{\mu_r \mu_{r-1}(\mathbf{V})}^G)_r(a; \mathbf{Y}) = a^{\Lambda_{r-2}} Y_{2,r}.$$

(ii) For $k \in [1, r-2]$, the cluster variables $(\varphi_{\mu_k \mu_{k+1} \cdots \mu_{r-3} \mu_{r-2} \mu_r \mu_{r-1}(\mathbf{V})}^G)_k(a; \mathbf{Y})$ in $\mathbb{C}[G^{e, c^2}]$ are described as

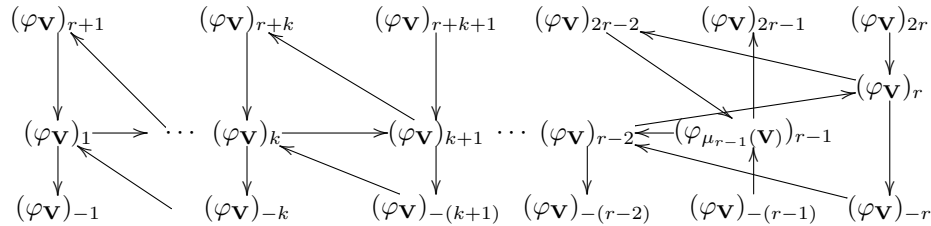
$$(\varphi_{\mu_k \mu_{k+1} \cdots \mu_{r-3} \mu_{r-2} \mu_r \mu_{r-1}(\mathbf{V})}^G)_k(a; \mathbf{Y}) = a^{\Lambda_{k-1} + \Lambda_{r-2}} Y_{2,k}.$$

[Proof.]

(i) The mutation diagram (7.39) and the equation (7.40) say that

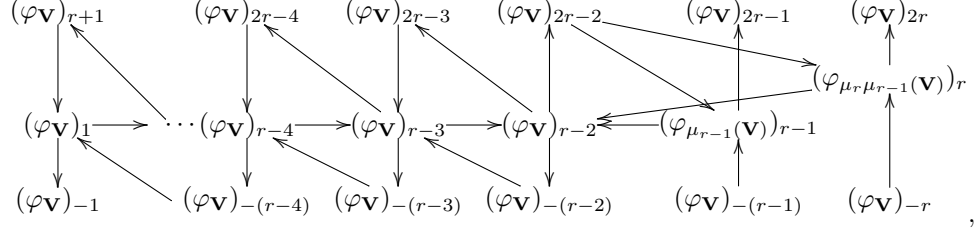
$$\begin{aligned} (\varphi_{\mu_{r-1}(\mathbf{V})}^G)_{r-1} &= \frac{(\varphi_{\mathbf{V}}^G)_{2r-2} (\varphi_{\mathbf{V}}^G)_{-(r-1)} + (\varphi_{\mathbf{V}}^G)_{r-2} (\varphi_{\mathbf{V}}^G)_{2r-1}}{(\varphi_{\mathbf{V}}^G)_{r-1}} \\ &= a^{\Lambda_{r-1} + \Lambda_{r-2}} \frac{Y_{1,r-1} Y_{2,r-1} + Y_{1,r-2} (1 + A_{1,r-2}^{-1} + \cdots + A_{1,r-2}^{-1} \cdots A_{1,1}^{-1})}{a^{\Lambda_{r-1}} Y_{1,r-1} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,1}^{-1})} \\ &= a^{\Lambda_{r-2}} Y_{2,r-1}. \end{aligned}$$

Next, the mutation diagram of $\mu_{r-1}(\Sigma_0)$ is as follows by Lemma 4.11:



Hence, by a calculation similar to the one of $(\varphi_{\mu_{r-1}}^G(\mathbf{V}))_{r-1}$, we obtain $(\varphi_{\mu_r \mu_{r-1}}^G(\mathbf{V}))_r(a; \mathbf{Y}) = a^{\Lambda_{r-2}} Y_{2,r}$.

(ii) We use the induction on $(r-2-k)$. First, let $r-2-k=0$, so that $k=r-2$. The mutation diagram of $\mu_r \mu_{r-1}(\Sigma_0)$ is

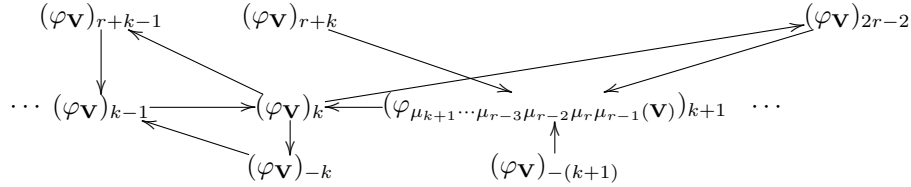


which yields

$$\begin{aligned}
& (\varphi_{\mu_{r-2} \mu_r \mu_{r-1}}^G(\mathbf{V}))_{r-2} \\
&= \frac{(\varphi_{\mu_r \mu_{r-1}}^G(\mathbf{V}))_r (\varphi_{\mu_{r-1}}^G(\mathbf{V}))_{r-1} (\varphi_{\mathbf{V}}^G)_{r-3} + (\varphi_{\mathbf{V}}^G)_{2r-3} (\varphi_{\mathbf{V}}^G)_{2r-2} (\varphi_{\mathbf{V}}^G)_{-(r-2)}}{(\varphi_{\mathbf{V}}^G)_{r-2}} \\
&= \frac{a^{2\Lambda_{r-2} + \Lambda_{r-3}} Y_{2,r-1} Y_{2,r} Y_{1,r-3} (1 + A_{1,r-3}^{-1} + \cdots + A_{1,r-3}^{-1} \cdots A_{1,1}^{-1}) + a^{2\Lambda_{r-2} + \Lambda_{r-3}} Y_{1,r-2} Y_{2,r-2}}{a^{\Lambda_{r-2}} Y_{1,r-2} (1 + A_{1,r-2}^{-1} + A_{1,r-2}^{-1} A_{1,r-3}^{-1} + \cdots + A_{1,r-2}^{-1} \cdots A_{1,1}^{-1})} \\
&= a^{\Lambda_{r-2} + \Lambda_{r-3}} Y_{2,r-2},
\end{aligned}$$

where we use (7.40) in the second equality.

Next, let $r-2-k > 0$. The vertices and arrows around the vertex $(\varphi_{\mathbf{V}}^G)_k$ in the mutation diagram of $\mu_{k+1} \mu_{k+2} \cdots \mu_{r-3} \mu_{r-2} \mu_r \mu_{r-1}(\Sigma_0)$ are



Hence,

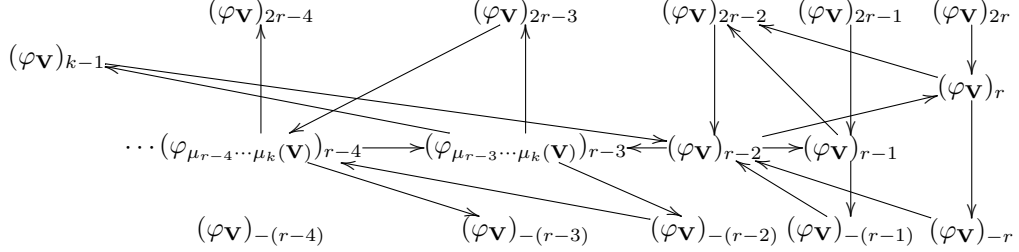
$$\begin{aligned}
& (\varphi_{\mu_k \mu_{k+1} \mu_{k+2} \cdots \mu_{r-3} \mu_{r-2} \mu_r \mu_{r-1}}^G(\mathbf{V}))_k \\
&= \frac{(\varphi_{\mu_{k+1} \mu_{k+2} \cdots \mu_{r-3} \mu_{r-2} \mu_r \mu_{r-1}}^G(\mathbf{V}))_{k+1} (\varphi_{\mathbf{V}}^G)_{k-1} + (\varphi_{\mathbf{V}}^G)_{r+k-1} (\varphi_{\mathbf{V}}^G)_{2r-2} (\varphi_{\mathbf{V}}^G)_{-k}}{(\varphi_{\mathbf{V}}^G)_k} \\
&= \frac{a^{\Lambda_{k-1} + \Lambda_k + \Lambda_{r-2}} (Y_{2,k+1} Y_{1,k-1} (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,1}^{-1}) + Y_{1,k} Y_{2,k})}{a^{\Lambda_k} Y_{1,k} (1 + A_{1,k}^{-1} + \cdots + A_{1,k}^{-1} \cdots A_{1,1}^{-1})} \\
&= a^{\Lambda_{k-1} + \Lambda_{r-2}} Y_{2,k},
\end{aligned}$$

where we use the induction hypothesis and (7.40) in the second equality. \square

[Proof of Theorem 6.5 (ii) and (iii).]

The claim (ii) is obtained by the same calculation as in Theorem 6.3 (ii). So let us consider the claim (iii).

First, by Lemma 4.11, the mutation diagram of $\mu_{r-3} \cdots \mu_{k+1} \mu_k(\Sigma_0)$ is



From this diagram,

$$\begin{aligned} & (\varphi_{\mu_{r-2} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-2}(a; \mathbf{Y}) \\ &= \frac{(\varphi_{\mathbf{V}}^G)_{k-1}(\varphi_{\mathbf{V}}^G)_{2r-2}(\varphi_{\mathbf{V}}^G)_{-(r-1)}(\varphi_{\mathbf{V}}^G)_{-r} + (\varphi_{\mu_{r-3} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-3}(\varphi_{\mathbf{V}}^G)_{r-1}(\varphi_{\mathbf{V}}^G)_r}{(\varphi_{\mathbf{V}}^G)_{r-2}}. \end{aligned} \quad (7.43)$$

Using (6.20) for $l = r - 3$, we obtain

$$\begin{aligned} & (\varphi_{\mu_{r-3} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-3} \\ &= a^{\Lambda_{k-1} + \Lambda_{r-2}} Y_{2,k} Y_{1,r-2} (1 + A_{1,r-2}^{-1} + A_{1,r-2}^{-1} A_{1,r-3}^{-1} + \cdots + A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \\ &= a^{\Lambda_{k-1}} Y_{2,k} (\varphi_{\mathbf{V}}^G)_{r-2} - a^{\Lambda_{r-2}} Y_{2,r-1} Y_{2,r} (\varphi_{\mathbf{V}}^G)_{k-1}, \end{aligned}$$

where we use (7.40) in the second equality. Similarly, using (7.40) and (7.41), we also get

$$(\varphi_{\mathbf{V}}^G)_{r-1} = a^{\Lambda_{r-1}} Y_{1,r-1} + a^{\Lambda_{r-1} - \Lambda_{r-2}} \frac{(\varphi_{\mathbf{V}}^G)_{r-2}}{Y_{2,r-1}}, \quad (\varphi_{\mathbf{V}}^G)_r = a^{\Lambda_r} Y_{1,r} + a^{\Lambda_r - \Lambda_{r-2}} \frac{(\varphi_{\mathbf{V}}^G)_{r-2}}{Y_{2,r}}.$$

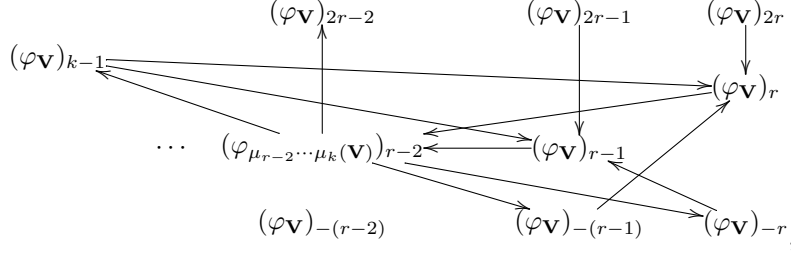
Applying these formulas to (7.43), it follows

$$\begin{aligned} & (\varphi_{\mu_{r-2} \cdots \mu_{k+1} \mu_k}^G(\mathbf{V}))_{r-2}(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1}} Y_{2,k} (\varphi_{\mathbf{V}}^G)_{r-1} (\varphi_{\mathbf{V}}^G)_r \\ & \quad - a^{\Lambda_{r-1} + \Lambda_r} Y_{2,r-1} Y_{2,r} (\varphi_{\mathbf{V}}^G)_{k-1} \left(\frac{Y_{1,r-1}}{Y_{2,r}} + \frac{Y_{1,r}}{Y_{2,r-1}} + a^{-\Lambda_{r-2}} \frac{(\varphi_{\mathbf{V}}^G)_{r-2}}{Y_{2,r-1} Y_{2,r}} \right) \quad (7.44) \\ &= a^{\Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{2,k} Y_{1,r-1} Y_{1,r} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,1}^{-1}) \\ & \quad \times (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,1}^{-1}) \\ & \quad - a^{\Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{1,k-1} Y_{2,r-1} Y_{2,r} (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,1}^{-1}) \\ & \quad \times \left(\frac{Y_{1,r}}{Y_{2,r-1}} + \frac{Y_{1,r-1}}{Y_{2,r}} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,1}^{-1}) \right) \\ &= a^{\Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{2,k} Y_{1,r-1} Y_{1,r} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,1}^{-1}) \\ & \quad \times (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \quad (7.45) \\ & \quad - a^{\Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{1,k-1} Y_{1,r} Y_{2,r} (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} \cdots A_{1,1}^{-1}) \\ &= a^{\Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{2,k} Y_{1,r-1} Y_{1,r} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,1}^{-1}) \\ & \quad \times (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \\ & \quad + a^{\Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{1,k-1} Y_{1,r-2} (1 + A_{1,r-2}^{-1} + \cdots + A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}) \\ & \quad \times (1 + A_{1,k-1}^{-1} + A_{1,k-1}^{-1} A_{1,k-2}^{-1} + \cdots + A_{1,k-1}^{-1} A_{1,k-2}^{-1} \cdots A_{1,1}^{-1}). \end{aligned}$$

By this explicit formula, the conclusion (6.23) follows.

[Proof of Theorem 6.5 (iv)]

The vertices and arrows around $(\varphi_{\mathbf{V}})_{r-1}$ in the mutation diagram of $\mu_{r-2}\mu_{r-3}\cdots\mu_{k+1}\mu_k(\Sigma_0)$ are



which yields

$$(\varphi_{\mu_{r-1}\mu_{r-2}\cdots\mu_{k+1}\mu_k(\mathbf{V})}^G)_{r-1}(a; \mathbf{Y}) = \frac{(\varphi_{\mu_{r-2}\cdots\mu_{k+1}\mu_k(\mathbf{V})}^G)_{r-2} + (\varphi_{\mathbf{V}}^G)_{k-1}(\varphi_{\mathbf{V}}^G)_{2r-1}(\varphi_{\mathbf{V}}^G)_{-r}}{(\varphi_{\mathbf{V}}^G)_{r-1}}. \quad (7.46)$$

It follows from (7.40) and (7.45) that

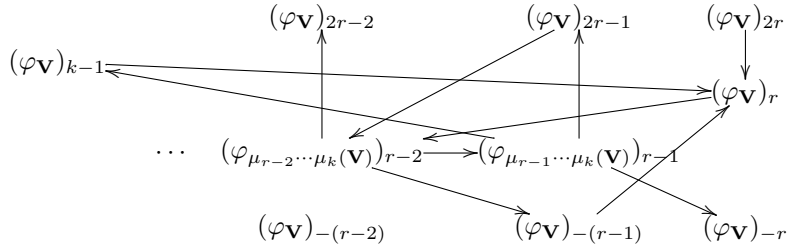
$$\begin{aligned} & (\varphi_{\mu_{r-2}\cdots\mu_{k+1}\mu_k(\mathbf{V})}^G)_{r-2}(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1}+\Lambda_r} Y_{2,k} Y_{1,r} (\varphi_{\mathbf{V}}^G)_{r-1} (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \\ & \quad - a^{\Lambda_{r-1}+\Lambda_r} Y_{1,r} Y_{2,r} (\varphi_{\mathbf{V}}^G)_{k-1}. \end{aligned} \quad (7.47)$$

Substituting this for (7.46), we see that

$$\begin{aligned} & (\varphi_{\mu_{r-1}\mu_{r-2}\cdots\mu_{k+1}\mu_k(\mathbf{V})}^G)_{r-1}(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1}+\Lambda_r} Y_{2,k} Y_{1,r} (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}), \end{aligned}$$

which implies (6.25).

Finally, we consider the cluster variable $(\varphi_{\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k(\mathbf{V})}^G)_r(a; \mathbf{Y})$. The mutation diagram of $\mu_{r-1}\mu_{r-2}\mu_{r-3}\cdots\mu_{k+1}\mu_k(\Sigma_0)$



implies that

$$(\varphi_{\mu_r\mu_{r-1}\mu_{r-2}\cdots\mu_{k+1}\mu_k(\mathbf{V})}^G)_r(a; \mathbf{Y}) = \frac{(\varphi_{\mu_{r-2}\cdots\mu_{k+1}\mu_k(\mathbf{V})}^G)_{r-2} + (\varphi_{\mathbf{V}}^G)_{k-1}(\varphi_{\mathbf{V}}^G)_{2r}(\varphi_{\mathbf{V}}^G)_{-(r-1)}}{(\varphi_{\mathbf{V}}^G)_r}. \quad (7.48)$$

From (7.44), by the same way as in (7.47), we can prove the following:

$$\begin{aligned} & (\varphi_{\mu_{r-2}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{r-2}(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1}+\Lambda_{r-1}} Y_{2,k} Y_{1,r-1} (\varphi_{\mathbf{V}}^G)_r (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}) \\ & \quad - a^{\Lambda_{r-1}+\Lambda_r} Y_{1,r-1} Y_{2,r-1} (\varphi_{\mathbf{V}}^G)_{k-1}. \end{aligned}$$

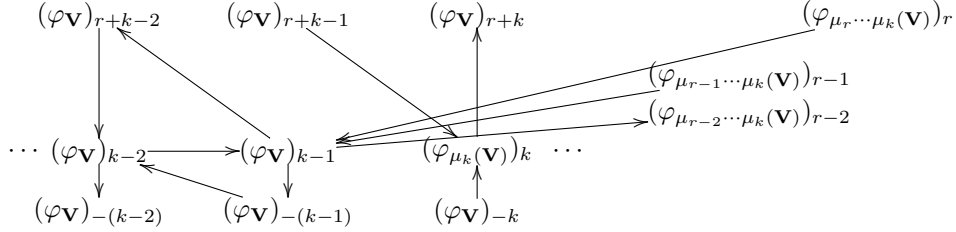
Substituting this for (7.48), we obtain

$$\begin{aligned} & (\varphi_{\mu_r\mu_{r-1}\mu_{r-2}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_r(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-1}+\Lambda_{r-1}} Y_{2,k} Y_{1,r-1} (1 + A_{1,r-1}^{-1} + A_{1,r-1}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}), \end{aligned}$$

which means (6.27). \square

[Proof of Theorem 6.5 (v).]

Using the induction on $(k-j)$, we shall prove (6.28). First, let $k-j = 1$. The vertices and arrows around $(\varphi_{\mathbf{V}})_{k-1}$ in the mutation diagram of $\mu_r\mu_{r-1}\mu_{r-2}\cdots\mu_{k+1}\mu_k(\Sigma_0)$ are



means that

$$\begin{aligned} & (\varphi_{\mu_{k-1}\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{k-1}(a; \mathbf{Y}) = \frac{1}{(\varphi_{\mathbf{V}}^G)_{k-1}} \left((\varphi_{\mu_{r-2}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{r-2} (\varphi_{\mathbf{V}}^G)_{r+k-2} (\varphi_{\mathbf{V}}^G)_{-(k-1)} \right. \\ & \quad \left. + (\varphi_{\mathbf{V}}^G)_{k-2} (\varphi_{\mu_{r-1}\mu_{r-2}\cdots\mu_k}^G(\mathbf{V}))_{r-1} (\varphi_{\mu_r\mu_{r-1}\cdots\mu_k}^G(\mathbf{V}))_r \right). \end{aligned} \quad (7.49)$$

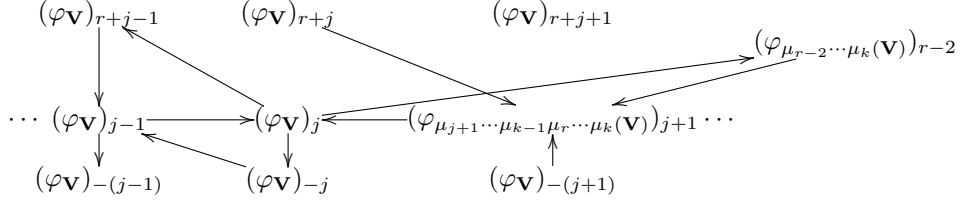
The explicit form (6.22) of $(\varphi_{\mu_{r-2}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{r-2}$ can be rewritten as

$$\begin{aligned} & (\varphi_{\mu_{r-2}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{r-2} \\ &= a^{-\Lambda_{k-1}} (\varphi_{\mu_{r-1}\mu_{r-2}\cdots\mu_k}^G(\mathbf{V}))_{r-1} (\varphi_{\mu_r\mu_{r-1}\cdots\mu_k}^G(\mathbf{V}))_r \frac{1}{Y_{2,k}} \\ & \quad + a^{\Lambda_{r-1}+\Lambda_r} Y_{1,r-2} (\varphi_{\mathbf{V}}^G)_{k-1} (1 + A_{1,r-2}^{-1} + A_{1,r-2}^{-1} A_{1,r-3}^{-1} + \cdots + A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}) \end{aligned} \quad (7.50)$$

by using (7.40), (6.24) and (6.26). Applying this and $(\varphi_{\mathbf{V}}^G)_{k-2} = a^{\Lambda_{k-2}-\Lambda_{k-1}} \frac{Y_{2,k-1}}{Y_{2,k}} (\varphi_{\mathbf{V}}^G)_{k-1} - a^{\Lambda_{k-2}} \frac{Y_{1,k-1} Y_{2,k-1}}{Y_{2,k}}$ (7.42) to (7.49), we obtain

$$\begin{aligned} & (\varphi_{\mu_{k-1}\mu_r\mu_{r-1}\cdots\mu_{k+1}\mu_k}^G(\mathbf{V}))_{k-1}(a; \mathbf{Y}) \\ &= a^{\Lambda_{k-2}-\Lambda_{k-1}} \frac{Y_{2,k-1}}{Y_{2,k}} (\varphi_{\mu_{r-1}\mu_{r-2}\cdots\mu_k}^G(\mathbf{V}))_{r-1} (\varphi_{\mu_r\mu_{r-1}\cdots\mu_k}^G(\mathbf{V}))_r \\ & \quad + a^{\Lambda_{k-2}+\Lambda_{k-1}+\Lambda_{r-1}+\Lambda_r} Y_{1,k-1} Y_{2,k-1} Y_{1,r-2} (1 + A_{1,r-2}^{-1} + \cdots + A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}) \\ &= a^{\Lambda_{k-2}+\Lambda_{k-1}+\Lambda_{r-1}+\Lambda_r} Y_{2,k-1} Y_{2,k} Y_{1,r-1} Y_{1,r} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} \cdots A_{1,k+1}^{-1}) \\ & \quad \times (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \\ & \quad + a^{\Lambda_{k-2}+\Lambda_{k-1}+\Lambda_{r-1}+\Lambda_r} Y_{1,k-1} Y_{2,k-1} Y_{1,r-2} (1 + A_{1,r-2}^{-1} + \cdots + A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}). \end{aligned}$$

Next, let us consider the case $k - j > 1$. The vertices and arrows around $(\varphi \mathbf{V})_j$ in the mutation diagram of $\mu_{j+1} \cdots \mu_{k-2} \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_{k+1} \mu_k(\Sigma_0)$ are as follows:



This diagram says that

$$\begin{aligned} & (\varphi_{\mu_j \mu_{j+1} \cdots \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_k(\mathbf{V})}^G)_j(a; \mathbf{Y}) \\ &= \frac{(\varphi_{\mu_{r-2} \cdots \mu_{k+1} \mu_k(\mathbf{V})}^G)_{r-2} (\varphi_{\mathbf{V}}^G)_{r+j-1} (\varphi_{\mathbf{V}}^G)_{-j} + (\varphi_{\mu_{j+1} \cdots \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_k(\mathbf{V})}^G)_{j+1} (\varphi_{\mathbf{V}}^G)_{j-1}}{(\varphi_{\mathbf{V}}^G)_j}. \end{aligned} \quad (7.51)$$

It follows by induction hypothesis that

$$\begin{aligned} & (\varphi_{\mu_{j+1} \cdots \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_k(\mathbf{V})}^G)_{j+1} \\ &= a^{\Lambda_j - \Lambda_{k-1}} (\varphi_{\mu_{r-1} \mu_{r-2} \cdots \mu_k(\mathbf{V})}^G)_{r-1} (\varphi_{\mu_r \mu_{r-1} \cdots \mu_k(\mathbf{V})}^G)_r \frac{Y_{2,j+1}}{Y_{2,k}} \\ &+ a^{\Lambda_j + \Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{2,j+1} Y_{1,k-1} Y_{1,r-2} (1 + A_{1,r-2}^{-1} + \cdots + A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \\ &\times (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} A_{1,k-2}^{-1} \cdots A_{1,j+2}^{-1}). \end{aligned}$$

Furthermore, we know that $(\varphi_{\mathbf{V}}^G)_{j-1} = a^{\Lambda_{j-1} - \Lambda_j} \frac{Y_{2,j}}{Y_{2,j+1}} (\varphi_{\mathbf{V}}^G)_j - a^{\Lambda_{j-1}} \frac{Y_{1,j} Y_{2,j}}{Y_{2,j+1}}$.

Thus, taking (7.50) and (7.51) into account, we obtain

$$\begin{aligned} & (\varphi_{\mu_j \mu_{j+1} \cdots \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_k(\mathbf{V})}^G)_j(a; \mathbf{Y}) = a^{\Lambda_{j-1} - \Lambda_j} \frac{Y_{2,j}}{Y_{2,j+1}} (\varphi_{\mu_{j+1} \cdots \mu_{k-1} \mu_r \mu_{r-1} \cdots \mu_k(\mathbf{V})}^G)_{j+1} \\ &+ \frac{a^{\Lambda_{j-1} + \Lambda_j + \Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r}}{(\varphi_{\mathbf{V}}^G)_j} Y_{1,j} Y_{2,j} Y_{1,r-2} Y_{1,k-1} \\ &\times (1 + A_{1,r-2}^{-1} + \cdots + A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}) A_{1,k-1}^{-1} \cdots A_{1,j+1}^{-1} (1 + A_{1,j}^{-1} + \cdots + A_{1,j}^{-1} \cdots A_{1,1}^{-1}) \\ &= a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{2,j} Y_{2,k} Y_{1,r-1} Y_{1,r} (1 + A_{1,r-1}^{-1} + \cdots + A_{1,r-1}^{-1} A_{1,r-2}^{-1} \cdots A_{1,k+1}^{-1}) \\ &\times (1 + A_{1,r}^{-1} + A_{1,r}^{-1} A_{1,r-2}^{-1} + \cdots + A_{1,r}^{-1} A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \\ &+ a^{\Lambda_{j-1} + \Lambda_{k-1} + \Lambda_{r-1} + \Lambda_r} Y_{2,j} Y_{1,k-1} Y_{1,r-2} (1 + A_{1,r-2}^{-1} + \cdots + A_{1,r-2}^{-1} A_{1,r-3}^{-1} \cdots A_{1,k+1}^{-1}) \\ &\times (1 + A_{1,k-1}^{-1} + \cdots + A_{1,k-1}^{-1} A_{1,k-2}^{-1} \cdots A_{1,j+1}^{-1}), \end{aligned}$$

which is our desired result. The description (6.29) immediately follows from (6.28). \square

References

- [1] A. Berenstein, S. Fomin, A. Zelevinsky, Cluster algebras III : Upper bounds and double Bruhat cells, Duke Math Journal, vol. 126 No.1, 1–52 (2005).

- [2] A. Berenstein, A. Zelevinsky, Tensor product multiplicities, canonical bases and totally positive varieties, *Invent. Math.* 143 No.1, 77–128 (2001).
- [3] S. Fomin, A. Zelevinsky, Double Bruhat cells and total positivity, *J. Amer. Math. Soc.*, vol.12, No.2, 335–380 (1999).
- [4] S. Fomin, A. Zelevinsky, Cluster algebras I: Foundations, *J. Amer. Math. Soc.*, vol.15, No.2, 497–529 (2002).
- [5] S. Fomin, A. Zelevinsky, Cluster algebras II: Finite type classification, *Invent. Math.* 154 No.1, 63–121 (2003).
- [6] C. Geiss, B. Leclerc, J. Schröer, Kac-Moody groups and cluster algebras, *Adv. Math.* 228 No.1, 329–433 (2011).
- [7] M. Gekhtman, M. Shapiro, A. Vainshtein, *Cluster Algebras and Poisson Geometry*, AMS (2010).
- [8] K. R. Goodearl, M. T. Yakimov, The Berenstein-Zelevinsky quantum cluster algebra conjecture, arXiv:1602.00498.
- [9] Y. Kanakubo, T. Nakashima, Cluster Variables on Certain Double Bruhat Cells of Type (u, e) and Monomial Realizations of Crystal Bases of Type A, *SIGMA*, vol.11, 1–32 (2015).
- [10] Y. Kanakubo, T. Nakashima, Cluster Variables on Double Bruhat Cells $G^{u,e}$ of Classical Groups and Monomial Realizations of Demazure Crystals, arXiv:1604.05956.
- [11] Y. Kanakubo, T. Nakashima, Cluster algebras of finite type via a Coxeter element and Demazure Crystals of type A, arXiv:1703.08323.
- [12] M. Kashiwara, Realizations of crystals, Combinatorial and geometric representation theory, *Contemporary Mathematics* 325, AMS, 133–139 (2003).
- [13] M. Kashiwara, Crystallizing the q -analogue of universal enveloping algebras, *Comm. Math. Phys.*, vol.133, 249–260 (1990).
- [14] M. Kashiwara, On crystal bases of the q -analogue of universal enveloping algebras, *Duke Math Journal* vol.63, No.2, 465–516 (1991).
- [15] M. Kashiwara, Bases cristallines des groupes quantiques, edited by Charles Cochet. *Cours Specialises*, 9, Societe Mathematique de France, Paris, (2002).
- [16] M. Kashiwara, T. Nakashima, Crystal graphs for representations of the q -analogue of classical Lie algebras, *J. Algebra*, vol.165, No.2, 295–345 (1994).
- [17] H. Nakajima, t -analogs of q -characters of quantum affine algebras of type A_n, D_n , *Contemp. Math.*, 325, AMS, Providence, RI, 141–160 (2003).
- [18] T. Nakashima, Decorations on Geometric Crystals and Monomial Realizations of Crystal Bases for Classical Groups, *J. Algebra*, vol.399, 712–769 (2014).